Differential Equations for Finance MA3607, 2013/14

Remedial Fourier series

Last year you would have covered some Fourier series material. Parts of this material will be used in this course. These notes are meant as a brief reminder - they are **not complete** and you should go over your notes and exercises from last year as well.

A very nice set of notes, exercises and videos on this can be found at
http://ocw.mit.edu/courses/mathematics/18-03sc-differential-equations-fall-2011/unit-iii-fourier-series-and-laplace-transform/fourier-series-basics/

A function $f(x)$, which is periodic with period $2L > 0$ satisfies

$$f(x+2L) = f(x)\,, \qquad \forall x \in \mathbf{R}\,, \tag{1}$$

and so if you know the values of the function in the domain $x \in [-L, L]$ then this periodicity property implies that you know the function's value for all $x \in \mathbf{R}$. Such a periodic function may be expressed in terms of its Fourier series, conventionally written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right) + b_n \sin\left(\frac{\pi n x}{L}\right)\,, \tag{2}$$

where a_n and b_n, the Fourier coefficients, are some numbers which will be different for different functions $f(x)$. Obtaining these coefficients is the main point of Fourier's theorem

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi n x}{L}\right) dx\,, \tag{3}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{\pi n x}{L}\right) dx\,. \tag{4}$$

N.B.: the above expression also determines the value of a_0 (and b_0) by setting $n = 0$ above ($b_0 \equiv 0$).

Example 1

Let $f(x)$ be a periodic function of x, with period $2L$, whose values in the range $x \in [-L, L]$ is $f(x) = \frac{x}{2L}$. A plot of this function is given in figure 1. Computing the Fourier series coefficients we find

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi n x}{L}\right) dx = \left[\frac{L^2\cos(\frac{n\pi x}{L})}{2n^2\pi^2} + \frac{Lx\sin(\frac{n\pi x}{L})}{2n\pi}\right]_{-L}^{L} = 0\,, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{\pi n x}{L}\right) dx = \left[\frac{L^2\sin(\frac{n\pi x}{L})}{2n^2\pi^2} - \frac{Lx\cos(\frac{n\pi x}{L})}{2n\pi}\right]_{-L}^{L} = \frac{(-1)^{n+1}L^2}{n\pi}\,, \end{aligned} \tag{5}$$

Can you explain why we should expect that $a_n = 0$ for all n without doing the integrals?

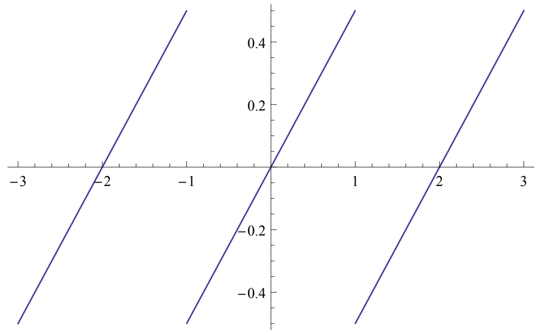

Figure 1: A plot of the function $f(x)$ from Example 1 for $L = 1$ in the range $[-3, 3]$

Exercises

1. Let $f(x)$ be a periodic function of x, with period $2L$, whose values in the range $x \in [-L, L]$ are given below; find its Fourier coefficients a_n and b_n

$$\begin{aligned} (i) &\quad f(x) = 5\,, \\ (ii) &\quad f(x) = 7 - \frac{3x}{L}\,, \\ (iii) &\quad f(x) = x^2\,, \\ (iv) &\quad f(x) = \sin\frac{6x}{L}\,, \\ (v) &\quad f(x) = \cos\frac{4x}{L}\,, \end{aligned}$$

2. With $L = 1$, draw the functions $f(x)$ defined above on the interval $x \in [-3, 3]$

Odd and even periodic functions

Recall that an even function $f(x)$ is defined as $\forall x,\ f(-x) = f(x)$; an odd function is defined as $\forall x,\ f(-x) = -f(x)$. If we consider a periodic function as above, and in addition say that in the range $x \in [-L, L]$ the function is even, then we see immediately that all the coefficients $b_n = 0$. Similarly, a periodic function odd in the range $x \in [-L, L]$ will have all coefficients $a_n = 0$. This is sensible, since cos and sin are, respectively, even and odd functions in the range $x \in [-L, L]$.

Exercises

3. Prove the above assertions about the vanishing of certain Fourier coefficients of periodic odd and even functions.

Fourier series for odd and even periodic functions

N.B.: Knowing the value of an even or odd periodic function (with period $2L$) in the range $[0, L]$ is sufficient to know its value for all x.

As we saw above an *odd* periodic function has a simplified Fourier series as compared to the general expression (2)

$$f_{\rm odd}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right) , \tag{6}$$

and because the function is odd on $[-L, L]$ the coefficients b_n can be determined in terms of integrals over the "half-period" $[0, L]$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n x}{L}\right) dx \, . \tag{7}$$

Similarly an *even* periodic function has a simplified Fourier series as compared to the general expression (2)

$$f_{\rm even}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right) , \tag{8}$$

and because the function is even on $[-L, L]$ the coefficients b_n can be determined in terms of integrals over the "half-period" $[0, L]$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi n x}{L}\right) dx \, . \tag{9}$$

Exercises

4. Show that for an odd periodic function obtaining the Fourier coefficients b_n using equation (7) gives the same result as using equation (4).
5. Show that for an even periodic function obtaining the Fourier coefficients a_n using equation (9) gives the same result as using equation (3).

Example 2

Let $f(x)$ be an odd periodic function of x, with period $2L$, whose values in the range $x \in [0, L]$ is $f(x) = \frac{x}{2L}$. From this we immediately conclude that this function is exactly the same as the function studied in Example 1. As a result, its Fourier coefficients are exactly the same as we computed in Example 1. Notice that as expected of an odd periodic function all $a_n = 0$. It is easy to check that calculating b_n using equation (7) gives the same result as the one obtained in Example 1.
Consider instead $f(x)$ an even periodic function of x, with period $2L$, whose values in the range $x \in [0, L]$ is $f(x) = \frac{x}{2L}$. A plot of this function is given in figure 2. Since this is an even periodic function we know that all the $b_n = 0$. The a_n Fourier coefficients are

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi n x}{L}\right) dx = \left[\frac{L^2\cos(\frac{n\pi x}{L})}{n^2\pi^2} + \frac{Lx\sin(\frac{n\pi x}{L})}{n\pi}\right]_0^L = \frac{(-1+(-1)^n)L^2}{n^2\pi^2} \tag{10}$$

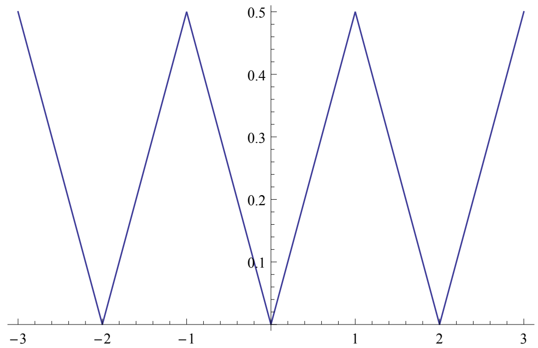

Figure 2: A plot of the even function $f(x)$ from Example 2 for $L = 1$ in the range $[-3, 3]$

Exercises

6. Consider $f(x)$ an **odd** function whose value on $[0, L]$ is given in exercise **1.** $(i) - (v)$. Determine the coefficients b_n of its Fourier series.
7. With $L = 1$, draw the functions $f(x)$ from the previous exercise for $x \in [-3, 3]$
8. Consider $f(x)$ an **even** function whose value on $[0, L]$ is given in exercise **1.** $(i) - (v)$. Determine the coefficients a_n of its Fourier series.
9. With $L = 1$, draw the functions $f(x)$ from the previous exercise for $x \in [-3, 3]$