Real Analysis: Exercise Sheet 1

1. There are two quantifiers: $\forall$ meaning 'for all' and $\exists$ meaning 'there exists'. Decide whether the following statements are true or false.

   (a) $\forall x \in \mathbb{R}, x^2 > 0$.
   (b) $\forall y \in \mathbb{R}, y^3 \leq 0$.
   (c) $\forall x \in \mathbb{R}, x^2 \geq 0$.
   (d) $\forall x, y \in \mathbb{R}, x^2 + y^2 > 0$.
   (e) $\forall x \in \mathbb{R}, x^2 < 0$.
   (f) $\exists x \in \mathbb{R}$ such that $x^2 \leq 0$.
   (g) $\exists x \in \mathbb{R}$ such that $x^2 + x + 1 < 0$.
   (h) $\exists x \in \mathbb{R}$ such that $x^2 + x + 1 > 0$.
   (i) $\exists x \in \mathbb{R}$ such that $x^3 < 0$.
   (j) $\exists y \in \mathbb{R}$ such that $y^3 \geq 0$.
   (k) $\forall y, z \in \mathbb{R}$, $(y^2 < z^2 \Rightarrow y < z)$.
   (l) $\forall x, y \in \mathbb{R}$, $(x < y \Rightarrow x^2 < y^2)$.
   (m) $\forall x \in \mathbb{R}$, $(x^3 > 0 \Rightarrow x > 0)$.
   (n) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $x > y$.
   (o) $\forall y \in \mathbb{R}$, $\exists x \in \mathbb{R}$ such that $x > y$.
   (p) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $y - x^2 > 1000$.
   (q) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$ $y \leq x$.
   (r) $\forall y \in \mathbb{R}$ $\exists x \in \mathbb{R}$ such that $x^2 < y$.
   (s) $\forall x \in \mathbb{R}$ $\exists y \in \mathbb{R}$ such that $xy \geq 0$.

2. Give a proof of the following statement.

$$(x^2 \text{ irrational} \Rightarrow x \text{ irrational})$$

   Is the converse true?

3. Decide whether the following statements are true or false.

   (a) x, y irrational $\Rightarrow x + y$ irrational.
   (b) x, y irrational $\Rightarrow xy$ irrational.

4. Prove that if $a > 0$ then $a^{-1} > 0$.

5. Prove that $\forall x, y \in \mathbb{R}$ we have

$$xy \leq \left(\frac{x+y}{2}\right)^2.$$