Real Analysis: Exercise Sheet 2

1. (a) Prove the following statement:

   $$\text{If } \forall \epsilon > 0 \text{ we have } a < b + \epsilon \text{ then } a \leq b.$$

   (Hint: Prove this by contradiction. Assume that $a > b$, then $a - b > 0$ so we can pick $\epsilon = a - b > 0$ then...)

   (b) Prove the following statement: Let $\epsilon > 0$. Then we have

   $$|x - a| < \epsilon \quad \Leftrightarrow \quad a - \epsilon < x < a + \epsilon.$$

   (Hint: There are two things to prove: $\Rightarrow$ and $\Leftarrow$. Use defintion of $|\ |$)

   (c) Use (a) and (b) to deduce that if $\forall \epsilon > 0$ $|x - a| < \epsilon$ then $x = a$.

2. Determine, if they exist, the maximum, minimum, supremum and infimum of the following sets.

   (a) $\{x \in \mathbb{R} : a \leq x \leq b\}$
   (b) $\{x \in \mathbb{R} : a < x \leq b\}$
   (c) $\{x \in \mathbb{R} : a \leq x < b\}$
   (d) $\{-3, 5, 0, 103, 59\}$
   (e) $\{1 + (-1)^n \frac{1}{n} : n \in \mathbb{N}\}$
   (f) $\{1 + (-1)^n n : n \in \mathbb{N}\}$
   (g) $\{x \in \mathbb{R} : 3x^2 - 10x + 3 < 0\}$
   (h) $\{\frac{1}{2^n} - \frac{1}{3^m} : n, m \in \mathbb{N}\}$
   (i) $\{x \in \mathbb{R} : |x - 1| < 2\}$

3. Show that given any element $x \in (0, \infty)$ there is another element $y \in (0, \infty)$ with the property that $y < x$. Deduce that $(0, \infty)$ has no minimum.

4. Prove that $S = \{\frac{n-1}{n} : n \in \mathbb{N}\}$ is bounded above and that its supremum is equal to 1. (Hint: First show that 1 is an upper bound for S. Then suppose, for a contradiction, that $\sup S = \alpha < 1$. Using corollary 1.3.2 from the lectures we can find a rational number $\alpha < \frac{p}{q} < 1$. Deduce that we can find an $n \in \mathbb{N}$ with $\alpha < \frac{n-1}{n}$, which contradicts the fact that α is an upper bound.)