Real Analysis: Exercise Sheet 3

1. Prove the following statements:
   (a) $\exists N \in \mathbb{N}$ such that $|\frac{n+3}{n} - 1| < 10^{-10}$ for all $n > N$.
   (b) $\exists N \in \mathbb{N}$ such that $|\frac{6n+3}{5n+1} - \frac{6}{5}| < 10^{-15}$ for all $n > N$.
   (c) $\exists N \in \mathbb{N}$ such that $|\frac{9n+2}{3n+7} - 3| < 10^{-14}$ for all $n > N$.

2. Use the definition of a convergent sequence to show the following:
   (a) $1 + \frac{1}{n} \to 1$ as $n \to \infty$.
   (b) $\frac{n^2-1}{n^2+1} \to 1$ as $n \to \infty$.
   (c) $\frac{n^2+n+1}{2n^2+1} \to \frac{1}{2}$ as $n \to \infty$.

3. (a) Let (x_n) be the sequence defined by

$$x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -4 & \text{if } n \text{ is even} \end{cases}$$

   Explain why (x_n) does not converge to 1.

   (b) Let (x_n) be the sequence defined by $x_n = n$ for all $n = 1, 2, 3, ...$ Explain why (x_n) does not converge to any limit l.

   (c) Let (x_n) be the sequence defined by

$$x_n = \begin{cases} 5 & \text{if } n \text{ is divisible by } 5 \\ \frac{1}{n} & \text{otherwise} \end{cases}$$

   Explain why (x_n) does not converge to 0.

4. (a) Give an example of a sequence in $[1, 2)$ which converges to 2.
   (b) Give an example of a sequence in $[1, 2)$ which diverges.
   (c) Give a example of a sequence of real numbers which is neither bounded above nor bounded below.
   (d) Give an example of a divergent sequence which is bounded above.