Real Analysis: Exercise Sheet 4

1. Use the Combination Theorem to show the following:
   (a) $\frac{3n^2+8n+1000}{n^3+2} \to 0$ as $n \to \infty$.
   (b) $\frac{n^8-1}{5n^8+4} \to \frac{1}{5}$ as $n \to \infty$.

2. (a) Show that for all $n \geq 1$ we have
$$\frac{n!}{n^n} \leq \frac{1}{n}.$$
   (b) Using (a) and the Sandwich rule, show that the sequence $(\frac{n!}{n^n})_{n=1}^{\infty}$ converges to 0.

3. (a) Give an example of a sequence which is increasing but not strictly increasing.
   (b) Give an example of a sequence which is decreasing and does not converge.
   (c) Give an example of a sequence which is strictly increasing and converges.

4. Consider the sequence
$$(x_n) = (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}).$$
   In this question we show that (x_n) is a convergent sequence. (In fact (x_n) converges to the real number e).
   (a) Prove that for all $n \geq 2$ we have $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$.
   (b) Using (a) and the fact that $1+a+a^2+\ldots+a^{n-1} = \frac{1-a^n}{1-a}$ to show that $2 < x_n < 3$ for all $n \geq 1$.
   (c) Show that (x_n) is an increasing sequence.
   (d) Deduce that (x_n) is convergent.

5. Consider the sequence (x_n) defined by
$$x_1 = \frac{5}{2}, \qquad 5x_{n+1} = x_n^2 + 6 \quad \text{for } n = 1, 2, 3, \ldots$$
   (a) Show by induction on n that $2 < x_n < 3$.
   (b) Using (a) show that $x_{n+1} - x_n < 0$ (so (x_n) is strictly decreasing).
   (c) Deduce that x_n is convergent and find its limit.