Real Analysis: Exercise Sheet 5

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ 1 + x & \text{for } 0 < x < 1 \\ 3 & \text{for } x = 1 \\ 2x^2 & \text{for } x > 1 \end{cases}$$

   Decide whether the following limits exist and if they do find their value.

   (a) $\lim_{x \to 0-} f(x)$
   (b) $\lim_{x \to 0+} f(x)$
   (c) $\lim_{x \to 0} f(x)$
   (d) $\lim_{x \to 1} f(x)$

   Hence determine where f is continuous and where it is not.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3 - x & \text{for } x > 1 \\ 1 & \text{for } x = 1 \\ 2x & \text{for } x < 1 \end{cases}$$

   Using the definitions of limits on the left and on the right to show that

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = 2.$$

   Is f continuous at $x = 1$?

3. (a) Prove that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$ (where x is measured in radians).
   (b) Use the identity $\cos x - \cos a = 2 \sin(\frac{x+a}{2}) \sin(\frac{a-x}{2})$ to show that

$$|\cos x - \cos a| \leq |x - a|.$$

   (c) Deduce that $f(x) = \cos x$ is continuous at every point a.

4. Using the Combination theorem and/or Sandwich rule for limits, show that the following limits exists and find their value.

   (a) $\lim_{x \to 0} \frac{x \sin x}{1+x^2}$
   (b) $\lim_{x \to 0} \frac{2x^2+1}{3x^2+3x+1}$
   (c) $\lim_{x \to 0} \frac{x \sin(1/x)}{1+x^2}$