Real Analysis: Exercise Sheet 6

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   (a) Show that f is continuous everywhere (you may use any standard result seen at the lecture).

   (b) Using the fact that $\lim_{t \to 0} \frac{\sin(t)}{t} = 1$, show that

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 1.$$

   (c) Sketch the graph of $f(x)$.

2. Use the Intermediate Value Theorem to show that

   (a) the equation $xe^{\sin(x)} = \cos(x)$ has a solution $x \in (0, \frac{\pi}{2})$.

   (b) the equation $2\sin(x) = x^2 - 1$ has a solution between 1 and 2.

   (c) the polynomial $17x^7 - 19x^5 - 1$ has a root between -1 and 0.

3. Decide whether the following statements are true or false. Justify your answers. (you may use any standard result seen at the lecture).

   (a) Every function $f : [0, 1] \to \mathbb{R}$ attains a maximum.

   (b) There exists a continuous function $f : (-2, 3] \to \mathbb{R}$ which is not bounded.

   (c) Every continuous function $f : [0, 1) \to \mathbb{R}$ which is bounded attains a maximum.

   (d) There exists a function $f : [-1, 1] \to [-1, 1]$ with no $x \in [-1, 1]$ satisfying $f(x) = x$.

   (e) Every function $f : [-1, 1] \to [-1, 1]$ is bounded.