Real Analysis: Exercise Sheet 7

1. Using the definition of a function being differentiable at a point show that the following functions are differentiable at every point $x = a$ with derivative given as shown.

   (a) $f(x) = k$ (a constant), $f'(a) = 0$.

   (b) $f(x) = x^n$, $f'(a) = na^{n-1}$.

   (c) $f(x) = \sin(x)$, $f'(a) = \cos(a)$.

   (d) $f(x) = \cos(x)$, $f'(a) = -\sin(a)$.

   You may use the fact that

   $$\sin(x) - \sin(a) = 2\cos(\frac{x+a}{2})\sin(\frac{x-a}{2}),$$

   $$\cos(x) - \cos(a) = 2\sin(\frac{x+a}{2})\sin(\frac{a-x}{2})$$

   and

   $$\lim_{t\to 0}\frac{\sin(t)}{t} = 1.$$

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} (x+1)^2 & x \geq 0 \\ \sin(2x) + 1 & x < 0 \end{cases}$$

   Show that $f(x)$ is continuous and differentiable everywhere and compute its derivative $f'(x)$. Is $f'(x)$ continuous everywhere? Is $f'(x)$ differentiable everywhere?

3. Let f, g be functions from (a, c) to $\mathbb{R}$ and let $b \in (a, c)$. Suppose that f and g are both differentiable at $x = b$ with derivative given by $f'(b)$ and $g'(b)$ respectively. Prove that the function $f + g$ is differentiable at $x = b$ with derivative given by $f'(b) + g'(b)$.

4. For each of the following functions, use the rules of differentiability (Combination theorem, Sandwich rule, Composition of functions) to prove that f is differentiable. In each case, find its derivative f'. You may use the fact that the functions 1, x, $\sin(x)$, $\cos(x)$ and e^x are differentiable and use their derivatives.

   (a) $f(x) = e^x\cos(x) + 1$

   (b) $f(x) = \frac{1+x^2}{1+x^4}$

   (c) $f(x) = \tan^3(x)$ $(x \neq \frac{\pi}{2} + n\pi)$

   (d) $f(x) = \begin{cases} x^2\cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$