Real Analysis: Exercise Sheet 8

1. Let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b). Use the Mean Value Theorem to prove
   (a) If $f'(x) < 0$ for all $x \in (a, b)$ then f is strictly decreasing on $[a, b]$.
   (b) If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$.
2. Suppose $f, g : [a, b] \to \mathbb{R}$ are functions which are continuous on $[a, b]$ and differentiable on (a, b). Prove that if $f'(x) = g'(x)$ for all $x \in (a, b)$ then $f(x) = g(x) +$ constant. (Hint: Use Question 1)
3. Use Rolle's Theorem to show that the polynomial $p(x) = x^3 + ax + b$ (with $a > 0$) has precisely one real root.
4. Use the Mean Value Theorem to prove that
   (a) $|\sin(b) - \sin(a)| \leq |b - a|$ for all $a, b \in \mathbb{R}$.
   (b) $\frac{1}{10} < \sqrt{83} - 9 < \frac{1}{9}$.
5. Decide whether the following statements are true or false. Justify your answers.
   (a) For all functions $f : [a, b] \to \mathbb{R}$ there exists a point $c \in (a, b)$ such that f is differentiable at c and $f'(c) = \frac{f(b)-f(a)}{b-a}$.
   (b) There exists a continuous function $f : [-1, 1] \to \mathbb{R}$ with no $x \in (-1, 1)$ such that f is differentiable at x and $f'(x) = \frac{f(1)-f(-1)}{2}$.
   (c) Let $f : (a, b) \to \mathbb{R}$ be a function which is continuous on (a, b) and differentiable at $c \in (a, b)$. If $f'(c) = 0$ then f has a local minimum or maximum at $x = c$.