Real Analysis: Coursework 2

This is an assessed coursework. Solutions should be handed in to the **mathematics general office** (CM326) by **4pm on Tuesday 17th April 2007**. Late submissions will be penalised.

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 + 2x - 3 & x < -1 \\ x^3 & -1 \leq x < 0 \\ 1 & x = 0 \\ \sin(x) & 0 < x < \frac{\pi}{2} \\ \frac{2x}{\pi} & x \geq \frac{\pi}{2} \end{cases}$$

   Determine whether the following limits exist and find them when they do exist.
   (a) $\lim_{x \to -1} f(x)$ (b) $\lim_{x \to 0} f(x)$ (c) $\lim_{x \to \frac{\pi}{2}} f(x)$
   Decide where the function is continuous and where it is not. (You may use any standard result seen at the lecture provided you state them clearly). [16]

2. Using the definition of continuity involving ϵ and δ, show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 7x - 4$ is continuous everywhere. [12]

3. Using the Intermediate Value Theorem, show that

   (a) the equation $2\ln(x) + \sqrt{x} - 2 = 0$ has a solution in the interval $[1, 2]$.
   (b) the polynomial $p(x) = -x^4 + 3x^3 + 1$ has at least two real roots.

   [12]

4. Using the Intermediate Value Theorem, prove the following statement:

   Let $f : [0, 1] \to \mathbb{R}$ be a continuous function satisfying $f(0) = f(1)$.
   Then there exists $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$.

   (Hint: consider the function $g(x) = f(x) - f(x + \frac{1}{2})$).
   Conclude that there are, at any time, antipodal points on the earth's equator that have the same temperature. [15]

5. Using the definition of differentiability, show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x^2 - 5x + 7$ is differentable at every point $x = b$ with derivative given by $f'(b) = 6b - 5$. [10]

6. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(x + 2) & x < -2 \\ x^2 + 5x + 6 & x \geq -2 \end{cases}$$

   is continuous and differentiable everywhere and compute $f'(x)$. Is $f'(x)$ continuous everywhere? Is $f'(x)$ differentiable everywhere?
   (You may use any standard result seen at lecture provided you state them clearly). [20]

7. Decide whether the following statements are true or false. Justify your answers.

   (a) There exists a function $f : [0, 1] \to \mathbb{R}$ which is continuous everywhere except at $x = \frac{1}{2}$.
   (b) All continuous functions $f : (2, 5] \to \mathbb{R}$ are bounded.
   (c) There exists a continuous function $f : [-1, 1) \to \mathbb{R}$ which is bounded but does not attain a minimum value on $[-1, 1)$.
   (d) For all functions $f : [-1, 1] \to [-1, 1]$ there exists some $x \in [-1, 1]$ satisfying $f(x) = x$.
   (e) There exists a continuous function $f : [0, 2] \to \mathbb{R}$ which is not differentiable at $x = 1$.

   [15]