Real Analysis: Revision questions

1. Define what is meant by 'a set S of real numbers is (i) bounded above, (ii) bounded below, (iii) bounded'.

2. Give examples of sets which are/are not bounded above/below.

3. Let S be a set of real numbers. Define what is meant by

   (a) 'the real number H is an upper/lower bound for the set S'.

   (b) 'the real number α is the maximum/minimum of the set S'.

   (c) 'the real number β is the supremum/infimum of the set S'.

4. State the continuum property.

5. Determine, if they exist, the maximum, minimum, supremum and infimum for the following sets of real numbers: $[3,5]$, $[3,5)$, $(3,5]$, $(3,5)$, $\{n+(-1)^n\frac{1}{n} : n \in \mathbb{N}\}$, $\{2+(-1)^n\frac{1}{n} : n \in \mathbb{N}\}$, $\{x \in \mathbb{R} : -x^2 - x + 6 < 0\}$, $\{x \in \mathbb{R} : 6x^2 + 17x - 3 < 0\}$, $\{x \in \mathbb{R} : 6x^2 + 17x - 3 \geq 0\}$, $\{\frac{1}{3^m} + \frac{1}{7^n} : n, m \in \mathbb{N}\}$, $\{\frac{1}{3^m} - \frac{1}{7^n} : n, m \in \mathbb{N}\}$, $\{\frac{1}{m} + \frac{1}{n} + \frac{1}{q} : m, n, q \in \mathbb{N}\}$, $\{x \in \mathbb{R} : |x+5| \leq 3\}$, $\{x \in \mathbb{R} : |x+5| < 3\}$, $\{x \in \mathbb{R} : |x+5| > 3\}$, $\{x \in \mathbb{R} : x^2 < 3\}$.

6. Decide whether the following statements are true or false. Justify your answers. You may use any standard results seen at the lecture provided you state them clearly.

   (a) Every set of real numbers has a minimum.

   (b) Every set of real numbers has a maximum.

   (c) Every set of real numbers which is bounded above has a maximum.

   (d) Every set of real numbers which is bounded below has a minimum.

   (e) If a set of real numbers has a minimum then it has an infimum.

   (f) If a set of real numbers has a maximum then it has a supremum.

   (g) There exists a set of real numbers with a supremum but no maximum.

   (h) There exists a set of real numbers with an infimum but no minimum.

   (i) There exists a set of real numbers with a maximum but no infimum.

7. Prove the following statements:

   (a) $[1, \infty)$ has no maximum.

   (b) $[-1, 7)$ has no maximum.

   (c) $(9, 16)$ has no minimum.

   (d) $(-\infty, -2]$ has no minimum.

(Hint: proof by contradiction)

8. Define what is meant by 'the sequence (x_n) converges to the limit l as n tends to infinity'.

9. Using the definition of a sequence converging to a limit, prove the following statements:

   (a) the sequence $\left(\frac{8n-3}{2n+1}\right)$ converges to 4 as $n \to \infty$.

   (b) the sequence $\left(\frac{8n+7}{3n+2}\right)$ converges to $\frac{8}{3}$ as $n \to \infty$.

   (c) the sequence (x_n) given by

   $$x_n = \begin{cases} 1 & \text{if } n \text{ is a prime} \\ 0 & \text{otherwise} \end{cases}$$

   does not converge to 0 as $n \to \infty$.

   (d) the sequence (x_n) given by

   $$x_n = \begin{cases} \frac{1}{5} & \text{if } n \text{ is divisible by 5} \\ 0 & \text{otherwise} \end{cases}$$

   does not converge to 0 as $n \to \infty$.

   (e) the sequence (x_n) given by

   $$x_n = \begin{cases} \frac{1}{n^2} & \text{if } n \text{ is even} \\ \frac{1}{2} & \text{if } n \text{ is odd} \end{cases}$$

   does not converge to 0 as $n \to \infty$.

10. Prove that if the sequence (x_n) converges to l as $n \to \infty$ and $\lambda \in \mathbb{R}$ then the sequence (λx_n) converges to λl as $n \to \infty$.

11. Prove that if the sequence (x_n) converges to l and the sequence (y_n) converges to m as $n \to \infty$ then the sequence $(x_n + y_n)$ converges to $l + m$ as $n \to \infty$.

12. Define what it means for a sequence of real numbers (x_n) to be increasing/decreasing.

13. When does an increasing/decreasing sequence converge?

14. Decide whether the following statements are true or false. Justify your answers. You may use any standard results seen at the lecture provided you state them clearly.

    (a) All sequences in $(0, \frac{1}{3})$ converge to a limit as $n \to \infty$.

    (b) There exists a decreasing sequence in $[0, 1]$ which is divergent.

    (c) There exists a sequence which is strictly decreasing and convergent.

    (d) All divergent sequences are unbounded.

    (e) If (x_n) is a sequence with $x_n < 5$ for all $n \in \mathbb{N}$ then (x_n) converges to 5.

    (f) If (x_n) is a sequence with $x_n < 5$ for all $n \in \mathbb{N}$ then (x_n) cannot converge to 6.

(g) There exists a sequence which is divergent and bounded above.

(h) There exists a sequence which is neither bounded below nor bounded above.

15. Decide whether or not the following sequences converge to a limit as $n \to \infty$. When they do, find their limit. You may use any standard result seen at the lecture provided you state them clearly.

    (a) $\left(\frac{5n^7+16n^5-3}{6n^7+3n^4+2n}\right)$

    (b) $\left(\frac{5n^2+3n+2}{2n+1}\right)$

    (c) $\left(\frac{2^{n-1}}{n!}\right)$

    (d) $\left(\frac{\sin(n)}{n}\right)$

    (e) $\left(\frac{\cos(n)}{n^2}\right)$

16. Consider the sequence (x_n) defined by $x_1 = 2$ and $6x_{n+1} = x_n^2 + 5$ for $n \geq 1$.

    (a) Show that for all n we have $1 < x_n < 5$.

    (b) Show that (x_n) is a decreasing sequence.

    (c) Deduce that (x_n) is convergent and find its limit.

    Same question for the sequence defined by $x_1 = 5$ and $9x_{n+1} = x_n^2 + 14$.

17. Define what it means for a function $f : (a, c) \to \mathbb{R}$ to be continuous at a point $b \in (a, c)$ using (a) limits, (b) ϵ and δ.

18. Using 17(b) show that the function $f(x) = 3x + 5$ is continuous everywhere.

19. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin(x) & x < 0 \\ 1 & x = 0 \\ x^2 & 0 < x < 1 \\ 2x - 1 & x \geq 1 \end{cases}$$

    Decide whether the limits $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist and determine where the function f is continuous and where it is not.

20. State the boundedness and extremum property for continuous functions $f : [a, b] \to \mathbb{R}$.

21. State the intermediate value theorem.

22. State the fixed point theorem.

23. Use the intermediate value theorem to prove the fixed point theorem. (hint: consider $g(x) = f(x) - x$)

24. Decide whether the following statements are true or false. Justify your answers. You may use any standard results from the lecture provided you state them clearly.

   (a) There exists a function $f : [0, 1] \to [0, 1]$ with no $x \in [0, 1]$ satisfying $f(x) = x$.
   (b) For all continuous functions $f : (0, 1) \to (0, 1)$ we can find $x \in (0, 1)$ with $f(x) = x$.
   (c) All continuous functions $f : [0, 1) \to \mathbb{R}$ attain a maximum value.
   (d) All continuous functions $f : (0, 1] \to \mathbb{R}$ attain a maximum value.
   (e) All continuous functions $f : (0, 1] \to \mathbb{R}$ attain a minimum value.
   (f) All continuous functions $f : [0, 1) \to \mathbb{R}$ attain a minimum value.
   (g) There exists a continuous function $f : (-3, 2] \to \mathbb{R}$ which is not bounded.
   (h) There exists a continuous function $f : (0, 1] \to \mathbb{R}$ which is bounded but does not attain a minimum value.
   (i) There exists a continuous function $f : (0, 1] \to \mathbb{R}$ which is bounded but does not attain a maximum value.
   (j) All functions $f : [-1, 1] \to \mathbb{R}$ are bounded.

25. Use the intermediate value theorem to show that

   (a) the equation $x = \cos(x)$ has a solution in the interval $[0, \frac{\pi}{2}]$.
   (b) the polynomial $p(x) = x^4 + 2x^3 - 2$ has at least two real roots.
   (c) the equation $e^{\cos(x)} = x \sin(x)$ has a solution in the interval $(0, \frac{\pi}{2})$.
   (d) the equation $3 \tan(x) = 2 + \sin(x)$ has a solution in the interval $(0, \frac{\pi}{4})$.

26. Define what it means for a function $f : (a, c) \to \mathbb{R}$ to be differentiable at a point $b \in (a, c)$.

27. Use the definition of differentiability at a point to show that the function $f(x) = 3x+5$ is differentiable at every point $x = b$ with derivative given by $f'(b) = 3$.

28. Use the definition of differentiability at a point to show that the function $f(x) = 3x^2+5x-6$ is differentiable at every point $x = b$ with derivative given by $f'(b) = 6b+5$.

29. Use the definition of differentiability to show that $f(x) = |x|$ (resp. $f(x) = |x - 5|$) is not differentiable at $x = 0$ (resp. at $x = 5$).

30. Let $f, g : (a, c) \to \mathbb{R}$ be two functions which are differentiable at the point $b \in (a, c)$. Show that the functions $f + g$ and λf (for $\lambda \in \mathbb{R}$) are also differentiable at $x = b$ with derivative given by $f'(b) + g'(b)$ and $\lambda f'(b)$ respectively.

31. Let $f : (a, c) \to \mathbb{R}$ be a function. Prove that if f is differentiable at point $b \in (a, c)$ then f is continuous at point b.

32. Give an example of a function which is continuous at a point but not differentiable at that point.

33. State Rolle's theorem.

34. State the mean value theorem.

35. Use the mean value theorem to prove the following statements:
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

   (a) If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing on $[a, b]$.
   (b) If $f'(x) < 0$ for all $x \in (a, b)$ then f is strictly decreasing on $[a, b]$.
   (c) If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$.

36. Using the mean value theorem prove the following

   (a) $\frac{1}{10} < \sqrt{83} - 9 < \frac{1}{9}$.
   (b) $\frac{3}{22} < \sqrt{103} - 10 < \frac{3}{20}$.
   (c) For $0 < a < b < \frac{\pi}{2}$ we have
$$(b - a)\cos(b) < \sin(b) - \sin(a) < (b - a)\cos(a).$$
   (d) For $0 < a < b < 1$ we have
$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}(b) - \sin^{-1}(a) < \frac{b-a}{\sqrt{1-b^2}}.$$
   (e) the polynomial $p(x) = x^3 + ax + b$ (with $a > 0$) has precisely one real root. (Hint: suppose for a contradiction that it had two real roots x_1 and x_2 then apply MVT on the interval $[x_1, x_2]$).

37. Decide whether the following statements are true or false. Justify your answers. You may use any standard results seen at the lecture provided you state them clearly.

   (a) There exists a function $f : [0, 2] \to \mathbb{R}$ with no $x \in (0, 2)$ such that f is differentiable at x and $f'(x) = \frac{f(2)-f(0)}{2}$.
   (b) For all continuous functions $f : [0, 1] \to \mathbb{R}$ we can find an $x \in (0, 1)$ such that f is differentiable at x and $f'(x) = f(1) - f(0)$.
   (c) Let $f : (2, 5) \to \mathbb{R}$ be a function which is continuous and differentiable on $(2, 5)$. If $f'(x) = 0$ for some $x \in (2, 5)$ then f has a local minimum or maximum at x.

38. Consider the following functions

   (a) $f(x) = \begin{cases} \cos(x) & x \geq 0 \\ x^2 + 1 & x < 0 \end{cases}$
   (b) $f(x) = \begin{cases} e^x - 1 & x \geq 0 \\ x & x < 0 \end{cases}$
   (c) $f(x) = \begin{cases} \cos(x - 3) & x \geq 3 \\ x^2 - 6x + 10 & x < 3 \end{cases}$

For each of the above function show that f is continuous and differentiable everywhere and determine its derivative $f'(x)$. Is $f'(x)$ continuous everywhere? Is $f'(x)$ differentiable everywhere?