1. (a) Let S be a set of real numbers. Define what is meant by

   i. S is bounded above.

   ii. the real number H is an upper bound for S.

   iii. the real number α is the supremum of S.

   iv. the real number β is the maximum of S.

   (b) Decide whether the following statements are true or false. Justify your answers.

   i. Every set of real numbers which is bounded above has a maximum.

   ii. If a set of real numbers has a maximum then it also has a supremum.

   iii. There exists a set of real numbers with a supremum but no maximum.

   (c) Find, if they exist, the supremum, infimum, maximum and minimum for the following sets of real numbers:

   i. $\{x \in \mathbb{R} : |x+6| \leq 2\}$

   ii. $\{\frac{1}{3^n} - \frac{1}{5^m}, : n, m \in \mathbb{N}\}$

   iii. $\{\frac{1}{3^n} + \frac{1}{5^m} : n, m \in \mathbb{N}\}$

   iv. $\{x \in \mathbb{R} : 5x^2 + 9x - 2 < 0\}$

   (d) Prove that the set $[1, 5)$ has no maximum.
   (Hint: Proof by contradiction).

Turn over ...

2. (a) Define what is meant by 'the sequence of real numbers (x_n) converges to a limit l as n tends to infinity'.
Use this definition to show that

   i. the sequence $(x_n) = \left(\frac{6n-5}{2n+3}\right)$ converges to 3 as n tends to infinity.
   ii. the sequence (x_n) defined by

$$x_n = \begin{cases} \frac{1}{3} & \text{if } n \text{ is divisible by } 3 \\ 0 & \text{otherwise} \end{cases}$$

   does not converge to 0.

(b) Consider the sequence (x_n) defined by

$$x_1 = 3 \qquad 7x_{n+1} = x_n^2 + 10 \quad \text{for } n \geq 1$$

   i. Show that $2 < x_n < 5$ for all $n \geq 1$.
   ii. Show that (x_n) is a decreasing sequence.
   iii. Deduce that (x_n) is convergent and find its limit.

(c) Decide whether the following statements are true or false. Justify your answers.

   i. All sequences in $(0, \frac{1}{2})$ are convergent.
   ii. There exists a convergent sequence which is strictly decreasing.
   iii. There exists a sequence which is neither bounded below nor bounded above.

3. (a) State the Intermediate Value Theorem.

   (b) Use it to prove the following statement:

   If a function $f : [a, b] \to [a, b]$ is continuous then there exists a point $x \in [a, b]$ such that $f(x) = x$.

   (Hint: consider the function $g(x) = f(x) - x$)

   (c) Using (a) and/or (b) show that

      i. the equation $x = \cos(x)$ has a solution in the interval $[0, \frac{\pi}{2}]$.

      ii. the polynomial $p(x) = x^4 + 2x^3 - 9$ has at least two real roots.

      iii. the equation $2\tan(x) = 1 + \cos(x)$ has a solution in the interval $(0, \frac{\pi}{4})$.

   (d) Decide whether the following statements are true or false. Justify your answers.

      i. There exists a function $f : [0, 1] \to [0, 1]$ with no $x \in [0, 1]$ satisfying $f(x) = x$.

      ii. For every continuous function $f : (0, 1) \to (0, 1)$ we can find $x \in (0, 1)$ satisfying $f(x) = x$.

      iii. Every continuous function $f : (0, 1] \to \mathbb{R}$ attains a maximum value for some $x \in (0, 1]$.

4. (a) Let $f : (a, c) \to \mathbb{R}$ be a function.

   i. Define what is meant by 'f is continuous at point $b \in (a, c)$'.

   ii. Define what is meant by 'f is differentiable at point $b \in (a, c)$'.

   iii. Prove that if f is differentiable at point b then f is continuous at point b.

   iv. Give an example of a function which is continuous at some point but not differentiable at that point.

   (b) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x + 1 & \text{if } x \leq 0 \\ e^x & \text{if } x > 0 \end{cases}$$

is continuous and differentiable everywhere and find its derivative $f'(x)$. Is $f'(x)$ continuous everywhere? Is $f'(x)$ differentiable everywhere?

(You may use the fact that the functions e^x and x are continuous and differentiable everywhere with derivative given by e^x and 1 respectively. You may also use any other standard results seen at the lecture provided you state them clearly).

5. (a) State the Mean Value Theorem.

(b) Use the Mean Value Theorem to prove the following statement:
Let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing on $[a, b]$.

(c) Use the Mean Value Theorem to show that

i.
$$\frac{1}{8} < \sqrt{51} - 7 < \frac{1}{7}$$

ii.
$$1 - \frac{a}{b} < \ln\frac{b}{a} < \frac{b}{a} - 1 \qquad (\text{for } 0 < a < b).$$

iii.
$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}} \qquad (\text{for } 0 < a < b < 1).$$

(You may use the fact that the function $\sqrt{x}$ is continuous and differentiable for $x > 0$ and use its derivative. You may also use the fact that the functions $\ln(x)$ and $\sin^{-1}(x)$ are continuous on $[a, b]$ and differentiable on (a, b) with derivative given by $\frac{1}{x}$ and $\frac{1}{\sqrt{1-x^2}}$ respectively.)

Internal Examiner:	Dr M. De Visscher
External Examiners:	Professor M.E. O'Neill
	Professor J. Billingham