Real Analysis: Solutions to Exercise Sheet 1

1. (a) False. take $x = 0$, then $x^2 = 0$.

   (b) False. Take $y = 1$, then $y^3 = 1$.

   (c) True. Let's prove it. Let $x \in \mathbb{R}$, then either $x > 0$ or $x = 0$ or $x < 0$. We consider these three cases separately. If $x > 0$ then $x^2 > 0.x = 0$ using (O3) (with $c = x > 0$), so in particular $x^2 \geq 0$. If $x = 0$ then $x^2 = 0 \geq 0$. Finally, if $x < 0$ then $x^2 > x.0 = 0$ using (O4) (with $c = x < 0$), in particular, $x^2 \geq 0$.

   (d) False. Take $x = y = 0$.

   (e) False. take $x = 1$.

   (f) True. There exists such an x, take $x = 0$.

   (g) False. The polynomial $x^2 + x + 1$ has no real roots, so the graph never crosses the x-axis. Moreover, for $x = 1$, $x^2 + x + 1 = 3 > 0$. Thus the graph is always above the x-axis, i.e. $x^2 + x + 1 > 0$ for all $x \in \mathbb{R}$. Hence, there is no x such that $x^2 + x + 1 < 0$.

   (h) True. Take $x = 1$.

   (i) True. Take $x = -1$.

   (j) True. Take $y = 1$.

   (k) False. Take $y = -1$ and $z = -2$ then $y^2 < z^2$ but $y > z$.

   (l) False. Take $x = -2$ and $y = -1$ then $x < y$ but $x^2 > y^2$.

   (m) True. We prove it by contradiction. Suppose $x^3 > 0$ and assume for a contradiction that $x \leq 0$. If $x = 0$ then $x^3 = 0$ but this is a contradiction. So we are left with the case $x < 0$. Using (O4) (with $c = x < 0$) we get $x^2 > x.0 = 0$. Using (O4) again (with $c = x < 0$), we get $x^3 < x.0 = 0$. This is a contradiction. Thus we must have $x > 0$.

   (n) True. For all x we can always take $y = x - 1$.

   (o) True. For all y we can always take $x = y + 1$.

   (p) True. For all x we can always take $y = 1007 + x^2$.

   (q) False. Suppose for a contradiction that such an x did exist, call it x_0. Then for all real numbers y, we should have $y \leq x_0$. In particular, this should be true for $y = x_0 + 1$ (which is a perfectly good real number). This means that we should have $x_0 + 1 \leq x_0$. But this is a contradiction.

   (r) False. Take $y = -1$ then there is no $x \in \mathbb{R}$ such that $x^2 < y$ as $x^2 \geq 0$ and $y = -1 < 0$.

   (s) True. For all x we can take $y = x$ then $xy = x^2 \geq 0$.

2. Proof by contrapositive i.e. it is equivalent to show that

$$(x \quad \text{rational} \quad \Rightarrow \quad x^2 \quad \text{rational})$$

Let $x = \frac{p}{q}$ where p and q are integers. Then $x^2 = \frac{p^2}{q^2}$ which is also rational. So we are done.

The converse says (x irrational $\Rightarrow x^2$ irrational). This is false as $x = \sqrt{2}$ is irrational but $x^2 = 2$ is rational.

3. (a) False. Take $x = 1 - \sqrt{2}$ and $y = 1 + \sqrt{2}$ then both x and y are irrational but $x + y = 2$ which is rational (i.e. not irrational!).

   (b) False. Take $x = y = \sqrt{2}$ then x and y are both irrational but $xy = 2$ which is rational.

4. Done at the lecture (corollary 1.1.1).

5. This is slightly more complicated to prove so let us first unpack the inequality to see what we get and then we'll write a rigourous proof.

$$xy \le \left(\frac{x+y}{2}\right)^2 = \frac{(x+y)^2}{4} = \frac{x^2 + 2xy + y^2}{4}$$

Thus, multiplying by 4 on both sides we get

$$4xy \le x^2 + 2xy + y^2$$

Substracting $4xy$ on both sides we get

$$0 \le x^2 - 2xy + y^2 = (x-y)^2$$

This is certainly true as the square of a real number of always positive or zero.
Now we write a proper proof (going backward, starting from what we know).
Using question 1(c), we have

$$(x-y)^2 \ge 0,$$

so

$$x^2 - 2xy + y^2 \ge 0$$

Now use (O2) (with $c = 4xy$) to add $4xy$ on both sides

$$x^2 + 2xy + y^2 \ge 4xy.$$

Thus we have

$$(x+y)^2 \ge 4xy.$$

Now use (O3) (with $c = \frac{1}{4} > 0$) to divide both sides by 4 and get

$$\frac{(x+y)^2}{4} \ge xy,$$

and so we get

$$\left(\frac{x+y}{2}\right)^2 \ge 0, \qquad \text{as required.}$$