Real Analysis: Solutions to Exercise Sheet 2

1. (a) Suppose, for a contradiction, that $a > b$, then $a - b > 0$. As $a < b + \epsilon$ for **all** $\epsilon > 0$, this should be true for $\epsilon = a - b$. So we must have

$$a < b + (a - b) = a.$$

But this is a contradiction (cannot have $a < a$). Thus our assumption was false and we must have $a \leq b$ as required.

(b) Start with '$\Rightarrow$'. We assume that $|x-a| < \epsilon$ and we want to prove that $a-\epsilon < x < a + \epsilon$. Either $x - a \geq 0$ or $x - a < 0$, we will consider these two cases separately. First suppose $x - a \geq 0$, then

$$x - a = |x - a| < \epsilon$$

and so adding a on both sides using (O2) we get $x < a + \epsilon$. On the other hand, as $x - a \geq 0$ and $\epsilon > 0$ we have $x - a > -\epsilon$. Adding a on both sides using (O2) we get $x > a - \epsilon$. So we are done in this case.

Next suppose $x - a < 0$, then

$$-(x - a) = |x - a| < \epsilon$$

and multiplying both sides by (-1) using (O4) we get $(x - a) > -\epsilon$. Now adding a on both sides using (O2) we get $x > a - \epsilon$. On the other hand $x - a < 0$ and $\epsilon > 0$ so we have $x - a < \epsilon$. Adding a on both sides we get $x < a + \epsilon$.

Now we turn to '$\Leftarrow$'. We assume that $a - \epsilon < x < a + \epsilon$ and we want to prove that $|x - a| < \epsilon$. Subtracting a on both sides (using (O2) with $c = -a$), we get

$$-\epsilon < x - a < \epsilon.$$

If $x - a \geq 0$ then $|x - a| = x - a < \epsilon$ so we are done. If $x - a < 0$ then $|x - a| = -(x - a)$. Now as $-\epsilon < x - a$ we get $\epsilon > -(x - a)$ (using (O4) with $c = -1$). Thus we get $|x - a| = -(x - a) < \epsilon$ as required.

(c) For all $\epsilon > 0$ we have $|x - a| < \epsilon$. Using (b) this implies that $x < a + \epsilon$ for all $\epsilon > 0$. But then using (a), we have that $x \leq a$. On the other hand, we also get from (b) that $a < x + \epsilon$ for all $\epsilon > 0$. So using (a) we get that $a \leq x$. We have proved that at the same time $a \leq x$ and $x \leq a$, so we must have $x = a$.

2. (a) This set is equal to $S = [a, b]$.
$\max S = b = \sup S$, $\min S = a = \inf S$.

(b) This set is equal to $S = (a, b]$.
$\max S = b = \sup S$, S has no minimum, $\inf S = a$.

(c) This set is equal to $S = [a, b)$.
S has no maximum, $\sup S = b$, $\min S = a = \inf S$.

(d) $\min S = -3 = \inf S$, $\max S = 103 = \sup S$.

(e) $\min S = 0 = \inf S$, $\max S = \frac{3}{2} = \sup S$.

(f) This set is not bounded above, not bounded below, it has no maximum, no minimum, no supremum and no infimum.

(g) The roots of the polynomial $3x^2 - 10x + 3$ are given by $\frac{1}{3}$ and 3 and the graph of the function $f(x) = 3x^2 - 10x + 3$ is 'facing upward'. Thus $3x^2 - 10x + 3 < 0$ precisely when $\frac{1}{3} < x < 3$. Thus this set can be rewritten as $(\frac{1}{3}, 3)$. So it has no minimum, no maximum, its infimum is $\frac{1}{3}$ and its supremum is 3.

(h) This set has no maximum, no minimum, its supremum is $\frac{1}{2}$ and its infimum is $-\frac{1}{3}$.

(i) Using Exercise 1(b) above, this set can be rewritten as

$$\{x \in \mathbb{R} \ : \ -1 < x < 3\} = (-1, 3),$$

so no maximum, no minimum, infimum equal to -1 and supremum equal to 3.

3. Let $x \in (0, \infty)$, then $x > 0$. We show that $y = \frac{x}{2}$ has the right property. Starting with the inequality $x > 0$ and using (O3) (with $c = \frac{1}{2}$), we get $\frac{x}{2} > 0$. Now adding $\frac{x}{2}$ on both sides (using (O2)) we get $x > \frac{x}{2}$. Thus $0 < \frac{x}{2} < x$ and we can take $y = \frac{x}{2} \in (0, \infty)$.

   Now we need to deduce that $(0, \infty)$ has no minimum. Suppose, for a contradiction, that it had a minimum $m \in (0, \infty)$. So $m > 0$, but by the above argument, we can find $y \in (0, \infty)$ (namely $y = \frac{m}{2}$) such that $y < m$. This contradicts the fact that m is a mimimum. Thus $(0, \infty)$ has no minimum.

4. First we show that 1 is an upper bound. We have $-1 < 0$ and adding n on both sides using (O2) we get $n - 1 < n$. Now dividing by n on both sides (i.e. using (O3) with $c = \frac{1}{n} > 0$) we get $\frac{n-1}{n} < 1$. This is true for all $n \in \mathbb{N}$, so 1 is an upper bound for S.

   We still need to show that 1 is the smallest upper bound. Suppose, for a contradiction, that it is not the smallest upper bound. Then S has a smaller upper bound α, say, with $\alpha < 1$. Using Corollary 1.3.2 from the lecture we can find a rational number between α and 1

$$\alpha < \frac{p}{q} < 1.$$

   As $\frac{p}{q} < 1$ we have $p \leq q - 1$ and so

$$\alpha < \frac{p}{q} \leq \frac{q-1}{q}.$$

   But this means that α is not an upper bound for S as if we take $n = q$ we have $\frac{n-1}{n} > \alpha$. This is a contradiction and so 1 is the smallest upper bound.