Real Analysis: Solutions to Exercise Sheet 3

1. (a) We need to find $N \in \mathbb{N}$ such that for all $n > N$ we have

$$|\frac{n+3}{n} - 1| < 10^{-10}.$$

So we want to know how large does n have to be in order to have

$$|\frac{n+3-n}{n}| = |\frac{3}{n}| = \frac{3}{n} < 10^{-10}.$$

This is equivalent to

$$n > 3.10^{10}.$$

Thus if we take $N = 3.10^{10}$ then for all $n > N$ we have

$$n > N = 3.10^{10}$$

and hence tracing the argument backwards we get that for all $n > N$ we have

$$|\frac{n+3}{n} - 1| < 10^{-10}$$

as required.

(b) We need to find $N \in \mathbb{N}$ such that for all $n > N$ we have

$$|\frac{6n+3}{5n+1} - \frac{6}{5}| < 10^{-15}.$$

So we want to know how large does n have to be in order to have

$$|\frac{6n+3}{5n+1} - \frac{6}{5}| = |\frac{9}{5(5n+1)}| = \frac{9}{5(5n+1)} < 10^{-15}.$$

This is equivalent to

$$n > \frac{1}{25}(9.10^{15} - 5).$$

Thus if we take N to be the smallest (or in fact any) integer greater or equal to $\frac{1}{25}(9.10^{15} - 5)$ then for all $n > N$ we have

$$n > N \geq \frac{1}{25}(9.10^{15} - 5)$$

and hence tracing the argument backwards we get that for all $n > N$ we have

$$|\frac{6n+3}{5n+1} - \frac{6}{5}| < 10^{-15}$$

as required.

(c) We need to find $N \in \mathbb{N}$ such that for all $n > N$ we have

$$|\frac{9n+2}{3n+7} - 3| < 10^{-14}.$$

So we want to know how large does n have to be in order to have

$$|\frac{9n+2}{3n+7} - 3| = |\frac{-19}{3n+7}| = \frac{19}{3n+7} < 10^{-14}.$$

This is equivalent to

$$n > \frac{1}{3}(19.10^{14} - 7).$$

Thus if we take N to be the smallest (or in fact any) integer greater or equal to $\frac{1}{3}(19.10^{14} - 7)$ then for all $n > N$ we have

$$n > N \geq \frac{1}{3}(19.10^{14} - 7)$$

and hence tracing the argument backwards we get that for all $n > N$ we have

$$|\frac{9n+2}{3n+7} - 3| < 10^{-14}$$

as required.

2. (a) Fix $\epsilon > 0$. We need to find $N \in \mathbb{N}$ such that for all $n > N$ we have

$$|(1 + \frac{1}{n}) - 1| < \epsilon.$$

So we want to know how large does n have to be in order to have

$$|(1 + \frac{1}{n}) - 1| = |\frac{1}{n}| = \frac{1}{n} < \epsilon.$$

This is equivalent to

$$n > \frac{1}{\epsilon}.$$

Thus if we take N to be the smallest (or in fact any) integer greater or equal to $\frac{1}{\epsilon}$ then for all $n > N$ we have

$$n > N \geq \frac{1}{\epsilon}$$

and hence tracing the argument backwards we get that for all $n > N$ we have

$$|(1 + \frac{1}{n}) - 1| < \epsilon$$

as required.

(b) Fix $\epsilon > 0$. We need to find $N \in \mathbb{N}$ such that for all $n > N$ we have

$$|\frac{n^2-1}{n^2+1} - 1| < \epsilon.$$

So we want to know how large does n have to be in order to have

$$|\frac{n^2-1}{n^2+1} - 1| = |\frac{-2}{n^2+1}| = \frac{2}{n^2+1} < \epsilon.$$

This is equivalent to

$$n^2 > \frac{2}{\epsilon} - 1.$$

Now either $\frac{2}{\epsilon} - 1 < 0$ and so the above equation is always satisfied, for any value of n and we can just take $N = 1$. Or $\frac{2}{\epsilon} - 1 \geq 0$. In this case if we take N to be the smallest (or in fact any) integer greater or equal to $\sqrt{\frac{2}{\epsilon} - 1}$ then for all $n > N$ we have

$$n > N \geq \sqrt{\frac{2}{\epsilon} - 1}$$

so for all $n > N$ we have

$$n^2 > \frac{2}{\epsilon} - 1$$

and hence tracing the argument backwards we get that for all $n > N$ we have

$$|\frac{n^2-1}{n^2+1} - 1| < \epsilon$$

as required.

(c) Fix $\epsilon > 0$. We want to find $N \in \mathbb{N}$ such that for all $n > N$ we have

$$|\frac{n^2+n+1}{2n^2+1} - \frac{1}{2}| < \epsilon.$$

So we want to know how large does n have to be in order to have

$$|\frac{n^2+n+1}{2n^2+1} - \frac{1}{2}| = |\frac{2n+1}{2(2n^2+1)}| = \frac{2n+1}{2(2n^2+1)} < \epsilon.$$

Note that

$$\frac{2n+1}{2(2n^2+1)} < \frac{2n+1}{2(2n^2)} < \frac{2n+n}{4n^2} = \frac{3}{4n}.$$

So if we have $\frac{3}{4n} < \epsilon$ then we also have $\frac{2n+1}{2(2n^2+1)} < \epsilon$. Now $\frac{3}{4n} < \epsilon$ is equivalent to

$$n > \frac{3}{4\epsilon}.$$

Thus if we take N to be the smallest (or in fact any) integer greater or equal to $\frac{3}{4\epsilon}$ then for all $n > N$ we have

$$n > N \geq \frac{3}{4\epsilon}.$$

Hence, tracing the argument backwards we have for all $n > N$

$$\frac{3}{4n} < \epsilon$$

and also

$$|\frac{n^2+n+1}{2n^2+1} - \frac{1}{2}| < \epsilon$$

as required.

3. In each of these exercises, we need to find one particular ϵ for which it is not possible to find an appropriate $N(\epsilon)$.

   (a) Take $\epsilon = 3$. We claim that it is not possible to find $N(3)$ such that for all $n > N(3)$ we have $|x_n - 1| < 3$. Indeed, whatever choice we make for $N(3)$ there will always be an even integer $n > N(3)$ with $|x_n - 1| = |-4-1| = 5 > 3$.

   (b) Take $\epsilon = 1$. We claim that it is not possible to find $N(1)$ such that for all $n > N(1)$ we have $|x_n - l| = |n - l| < 1$. Indeed, whatever choice we make for $N(1)$ there is always an integer n which is greater than $N(1)$ and also greater that $|l| + 1$. For such n we have

   $$|n - l| \geq |n| - |l| = n - |l| > 1$$

   (using Theorem 1.1.6).

   (c) Take $\epsilon = 2$. We claim that it is not possible to find $N(2)$ such that for all $n > N(2)$ we have $|x_n - 0| = x_n < 2$. Indeed, whatever choice we make for $N(2)$ there is always an integer $n > N(2)$ which is divisible by 5. For such an n we have

   $$|x_n - 0| = x_n = 5 > 2.$$

4. I give here one example for each questions but there are of course many other examples.

   (a) $(2 - \frac{1}{n})_{n=1}^{\infty}$.

   (b) $(1, \frac{3}{2}, 1, \frac{3}{2}, 1, \frac{3}{2}, 1, \frac{3}{2}, ...)$.

   (c) $((-1)^n n)_{n=1}^{\infty}$.

   (d) $(-n)_{n=1}^{\infty}$.