Real Analysis: Solutions to Exercise Sheet 4

1. (a) Dividing top and bottom by n^3 we get

$$\frac{3n^2+8n+1000}{n^3+2} = \frac{3\frac{1}{n}+8\frac{1}{n^2}+1000\frac{1}{n^3}}{1+2\frac{1}{n^3}}.$$

We know that $\frac{1}{n} \to 0$, $\frac{1}{n^2} \to 0$ and $\frac{1}{n^3} \to 0$ as $n \to \infty$, so using the Combination Theorem we see that

$$\frac{3n^2+8n+1000}{n^3+2} = \frac{3\frac{1}{n}+8\frac{1}{n^2}+1000\frac{1}{n^3}}{1+2\frac{1}{n^3}} \to \frac{3.0+8.0+1000.0}{1+2.0} = 0 \qquad \text{as } n \to \infty.$$

(b) Dividing top and bottom by n^8 we get

$$\frac{n^8-1}{5n^8+4} = \frac{1-\frac{1}{n^8}}{5+4\frac{1}{n^8}}.$$

we know that $\frac{1}{n^8} \to 0$ as $n \to \infty$, so using the Combination Theorem we see that

$$\frac{n^8-1}{5n^8+4} = \frac{1-\frac{1}{n^8}}{5+4\frac{1}{n^8}} \to \frac{1-0}{5+4.0} = \frac{1}{5} \qquad \text{as } n \to \infty.$$

2. (a)

$$\begin{aligned} \frac{n!}{n^n} &= \frac{n.(n-1).(n-2)...2.1}{n.n.n...n.n} \\ &= \frac{n}{n}.\frac{n-1}{n}.\frac{n-2}{n}...\frac{2}{n}.\frac{1}{n} \\ &\leq \frac{1}{n} \end{aligned}$$

as $\frac{n-i}{n} \leq 1$ for all $0 \leq i \leq n-1$.

(b) Using (a) we see that for all n we have

$$0 \leq \frac{n!}{n^n} \leq \frac{1}{n}.$$

As $\frac{1}{n} \to 0$ as $n \to \infty$, the Sandwich rule tells us that $\frac{n!}{n^n} \to 0$ as $n \to \infty$.

3. (a) $(1,1,1,1,...)$ is increasing but not strictly increasing. Another example would be $(1,1,1,2,3,4,5,6,....)$.

(b) $(-n)$ is decreasing but it does not converge (as it is not bounded).

(c) $(1-\frac{1}{n})$ is strictly increasing and bounded above (by 1), so it is convergent.

4. (a)

$$\begin{aligned}
\frac{1}{n!} &= \frac{1}{n}.\frac{1}{n-1}.\frac{1}{n-2}...\frac{1}{2}.\frac{1}{1} \\
&\leq \frac{1}{2}.\frac{1}{2}.\frac{1}{2}...\frac{1}{2}.1 \\
&= \frac{1}{2^{n-1}}.
\end{aligned}$$

(b) On the one hand

$$x_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ... + \frac{1}{n!} > 2.$$

On the other hand

$$\begin{aligned}
x_n &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ... + \frac{1}{n!} \\
&\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^{n-1}} \\
&= 1 + (1 + (\frac{1}{2}) + (\frac{1}{2})^2 + (\frac{1}{2})^3 + ... + (\frac{1}{2})^{n-1}) \\
&= 1 + \frac{1 - (\frac{1}{2})^n}{1 - (\frac{1}{2})} \\
&< 1 + \frac{1}{1 - \frac{1}{2}} = 1 + 2 = 3.
\end{aligned}$$

(c) For all $n \geq 1$ we have

$$x_{n+1} - x_n = \frac{1}{(n+1)!} > 0$$

so for all $n \geq 1$ we have

$$x_{n+1} > x_n$$

and the sequence is increasing.

(d) As the sequence is increasing and bounded above, using Theorem 2.3.1, we know that (x_n) is convergent.

5. (a) For $n = 1$ we have $2 < x_1 = \frac{5}{2} < 3$. Now let $n \geq 1$ and assume that $2 < x_n < 3$, we then need to prove that $2 < x_{n+1} < 3$. We have

$$2^2 + 6 < 5x_{n+1} = x_n^2 + 6 < 3^2 + 6,$$

so we get

$$10 < 5x_{n+1} < 15$$

and hence

$$2 < x_{n+1} < 3$$

as required.

(b) For all $n \geq 1$ we have

$$x_{n+1} - x_n = \frac{x_n^2 + 6}{5} - x_n = \frac{1}{5}x_n^2 - x_n + \frac{6}{5}.$$

Now the roots of this polynomial are 2 and 3 and the graph of this parabola is 'facing upwards'. Using (a) we know that $2 < x_n < 3$ and so in this case $\frac{1}{5}x_n^2 - x_n + \frac{6}{5} < 0$.

(c) As (x_n) is decreasing and bounded below, using Theorem 2.3.1 we know that it converges to a limit l say. We want to find l. First note that if $x_n \to l$ as $n \to \infty$ we also have $x_{n+1} \to l$ as $n \to \infty$. Now take the limit as $n \to \infty$ on both sides of the equation

$$5x_{n+1} = x_n^2 + 6.$$

Using the Combination Theorem we have

$$5x_{n+1} \to 5l \qquad \text{as } n \to \infty$$

and

$$x_n^2 + 6 \to l^2 + 6 \qquad \text{as } n \to \infty.$$

So we get

$$5l = l^2 + 6$$

and hence

$$l^2 - 5l + 6 = (l - 2)(l - 3) = 0.$$

Thus either $l = 2$ or $l = 3$. As $2 < x_n < 3$ and the sequence is strictly decreasing we must have $l = 2$.