Real Analysis: Solutions to Exercise Sheet 5

1. (a) $\lim_{x\to 0-} f(x) = 0$.
   (b) $\lim_{x\to 0+} f(x) = 1$.
   (c) $\lim_{x\to 0} f(x)$ does not exist as $\lim_{x\to 0-} \neq \lim_{x\to 0+}$.
   (d) $\lim_{x\to 1-} f(x) = \lim_{x\to 1-}(1+x) = 2$ and $\lim_{x\to 1+} f(x) = \lim_{x\to 1+} 2x^2 = 2$ so $\lim_{x\to 1} f(x)$ exists and is equal to 2.

   f is continuous everywhere (using Combination theorem) except at the point $x = 0$ (as $\lim_{x\to 0} f(x)$ does not exists) and at the point $x = 1$ (as $3 = f(1) \neq \lim_{x\to 1} f(x) = 2$).

2. Recall the definitions of limit on the left and limit on the right.

   We say that $\lim_{x\to b-} f(x) = l$ if $\forall \epsilon > 0$ we can find a $\delta > 0$ (which depends on ϵ) such that $\forall x$ with $b - \delta < x < b$ we have $|f(x) - l| < \epsilon$.

   We say that $\lim_{x\to b+} f(x) = l$ if $\forall \epsilon > 0$ we can find a $\delta > 0$ (which depends on ϵ) such that $\forall x$ with $b < x < b + \delta$ we have $|f(x) - l| < \epsilon$.

   Let us first prove that $\lim_{x\to 1-} f(x) = 2$.

   Fix $\epsilon > 0$. We need to find $\delta > 0$ (which depends on ϵ) such that whenever $1-\delta < x < 1$ we have $|f(x)-2| < \epsilon$. Now as x tends to 1 from the left we have $x < 1$ and so in this case $f(x) = 2x$. We want to have

$$|f(x) - 2| = |2x - 2| = 2|x - 1| < \epsilon.$$

   Take $\delta = \frac{\epsilon}{2}$. Then for x with $1 - \frac{\epsilon}{2} < x < 1$, we have $|x - 1| = 1 - x < \frac{\epsilon}{2}$ and so

$$|f(x) - 2| = 2|x - 1| < 2\frac{\epsilon}{2} = \epsilon$$

   as required.

   Now we prove that $\lim_{x\to 1+} f(x) = 2$.

   Fix $\epsilon > 0$. We need to find $\delta > 0$ (which depends on ϵ) such that whenever $1 < x < 1+\delta$ we have $|f(x) - 2| < \epsilon$. Now as x tends to 1 from the right we have $x > 1$ and so in this case $f(x) = 3 - x$. We want to have

$$|f(x) - 2| = |3 - x - 2| = |1 - x| < \epsilon.$$

   Take $\delta = \epsilon$. Then for x with $1 < x < 1 + \epsilon$, we have $|1 - x| = x - 1 < \epsilon$ and so

$$|f(x) - 2| = |1 - x| < \epsilon$$

   as required.

   As the limit on the left and on the right coincide we have that $\lim_{x\to 1} f(x)$ exists and is equal to 2. But as $f(1) = 1 \neq 2$ the function f is not continuous at the point $x = 1$.

3. (a) As $|\sin(x)| \leq 1$, the result is clear for $|x| \geq 1$. Thus we can certainly assume that $-\frac{\pi}{2} < -1 < x < 1 < \frac{\pi}{2}$. We will prove the result when $0 \leq x < \frac{\pi}{2}$ (the case $-\frac{\pi}{2} < x \leq 0$ is similar). Take a disc of radius 1. Then the area of the disc is π. and so the area of the portion of the disc defined by x is given by $\frac{x}{2}$.

Now the area of the triangle defined by the angle x (see above picture) is given by $\frac{\sin(x)}{2}$. This is always less or equal to the portion of the disc defined by x i.e. we have

$$\frac{\sin(x)}{2} \leq \frac{x}{2}.$$

But this implies

$$\sin(x) \leq x$$

as required.

(b)

$$\begin{aligned}
|\cos(x) - \cos(a)| &= |2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{a-x}{2}\right)| \\
&= 2|\sin\left(\frac{x+a}{2}\right)|.|\sin\left(\frac{a-x}{2}\right)| \\
&\leq 2|\sin\left(\frac{a-x}{2}\right)| \qquad \text{as } |\sin\left(\tfrac{x+a}{2}\right)| \leq 1 \\
&\leq 2|\frac{a-x}{2}| = |x-a| \qquad \text{using (a)}
\end{aligned}$$

(c) Recall the definition of continuity at a point a. We say that a function f is continuous at a point a if $\forall \epsilon > 0$, we can find a $\delta > 0$ (depending on ϵ) such that $\forall x$ with $|x-a| < \delta$ we have $|f(x) - f(a)| < \epsilon$.

Fix $\epsilon > 0$. We need to find a $\delta > 0$ such that whenever $|x-a| < \delta$ we have $|\cos(x) - \cos(a)| < \epsilon$.
Now note that using (b) we have

$$|\cos(x) - \cos(a)| \leq |x-a|.$$

So if we take $\delta = \epsilon$ then for all x's with $|x-a| < \delta = \epsilon$ we have

$$|\cos(x) - \cos(a)| \leq |x-a| < \epsilon$$

as required.

4. (a) First note that

$$\frac{-|x|}{1+x^2} \leq \frac{x\sin(x)}{1+x^2} \leq \frac{|x|}{1+x^2}.$$

Now using the Combination theorem we see that

$$\lim_{x\to 0} \frac{-|x|}{1+x^2} = 0$$

and

$$\lim_{x\to 0} \frac{|x|}{1+x^2} = 0.$$

Thus using the Sandwich rule we have

$$\lim_{x\to 0} \frac{x\sin(x)}{1+x^2} = 0.$$

(b) Using the combination theorem we see that

$$\lim_{x\to 0} \frac{2x^2+1}{3x^2+3x+1} = \frac{2.0+1}{3.0+3.0+1} = 1.$$

(c) First note that

$$\frac{-|x|}{1+x^2} \leq \frac{x\sin(1/x)}{1+x^2} \leq \frac{|x|}{1+x^2}.$$

So using the combination theorem and Sandwich rule just as in (a) we get

$$\lim_{x\to 0} \frac{x\sin(1/x)}{1+x^2} = 0.$$