Real Analysis: Solutions to Exercise Sheet 6

1. (a) The function x and $\sin(x)$ are continuous everywhere, using the Combination theorem we see that the function $\frac{1}{x}$ is continuous for $x \neq 0$ and hence using the Combination theorem and the Composition of function we see that $x\sin(\frac{1}{x})$ is continuous for $x \neq 0$.

   Now, we have

$$-|x| \leq x\sin(\frac{1}{x}) \leq |x| \qquad \forall x$$

   and

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0.$$

   Thus using the Sandwich Rule we get that

$$\lim_{x\to 0} x\sin(\frac{1}{x}) = 0 = f(0).$$

   Hence, $f(x)$ is continuous at $x = 0$.

   (b)

$$\begin{aligned}
\lim_{x\to+\infty} x\sin(\frac{1}{x}) &= \lim_{x\to+\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} \\
&= \lim_{t\to 0+} \frac{\sin(t)}{t} = 1, \\
\lim_{x\to-\infty} x\sin(\frac{1}{x}) &= \lim_{x\to-\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} \\
&= \lim_{t\to 0-} \frac{\sin(t)}{t} = 1.
\end{aligned}$$

2. (a) Consider the function $f(x) = xe^{\sin(x)} - \cos(x)$. Using the Combination theorem and Composition of function, $f(x)$ is continuous everywhere. Now, $f(0) = -1$ and $f(\frac{\pi}{2}) = \frac{\pi}{2}e > 0$. So we can apply the intermediate value theorem on $[0, \frac{\pi}{2}]$ which tells us that there exists $0 < c < \frac{\pi}{2}$ with $f(c) = 0$.

   (b) Consider the function $f(x) = 2\sin(x) - x^2 + 1$. Using the Combination theorem, $f(x)$ is continuous everywhere. Now, $f(1) = 2\sin(1) > 0$ and $f(2) = 2\sin(2) - 3 < 0$. So we can apply the intermediate value theorem on $[1, 2]$ to deduce that there exists $1 < c < 2$ with $f(c) = 0$.

   (c) Consider the function $f(x) = 17x^7 - 19x^5 - 1$. Using the Combination theorem, $f(x)$ is continuous everywhere. Now $f(-1) = -17 + 19 - 1 = 1 > 0$ and $f(0) = -1 < 0$. So we can apply the intermediate value theorem to deduce that there exists $-1 < c < 0$ with $f(c) = 0$.

3. (a) False. Take $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ 0 & x = 1 \end{cases}$$

then f does not attain a maximum value.

(b) True. Take $f : (-2, 3] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x+2}$ then f is continuous but not bounded.

(c) False. Take $f : [0, 1) \to \mathbb{R}$ defined by $f(x) = x$. Then f is continuous and bounded but does not attain a maximum.

(d) True. Take $f : [-1, 1] \to [-1, 1]$ defined by

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

then there is no $x \in [-1, 1]$ satisfying $f(x) = x$.

(e) False. Take $f : [-1, 1] \to [-1, 1]$ defined by $f(x) = \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Then f is not bounded.