Real Analysis: Solutions to Exercise Sheet 7

1. (a) Consider
$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{k-k}{x-a}=\lim_{x\to a}0=0.$$
So f is differentiable at $x=a$ with derivative given by $f'(a)=0$.

   (b) Consider
$$\begin{aligned}\lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{x^n-a^n}{x-a}\\ &= \lim_{x\to a}(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+a^{n-2}x+a^{n-1})\\ &= na^{n-1}.\end{aligned}$$
So f is differentiable at $x=a$ with derivative given by $f'(a)=na^{n-1}$.

   (c) Consider
$$\begin{aligned}\lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{\sin(x)-\sin(a)}{x-a}\\ &= \lim_{x\to a}\frac{2\cos(\frac{x+a}{2})\sin(\frac{x-a}{2})}{x-a}\\ &= \lim_{x\to a}\cos(\frac{x+a}{2})\ \frac{\sin(\frac{x-a}{2})}{\frac{x-a}{2}}\\ &= \lim_{x\to a}\cos(\frac{x+a}{2})\ \lim_{x\to a}\frac{\sin(\frac{x-a}{2})}{\frac{x-a}{2}}\\ &\qquad\text{(provided these limits exist)}\\ &= \cos(a).1\\ &= \cos(a).\end{aligned}$$
So f is differentiable at $x=a$ with derivative given by $f'(a)=\cos(a)$.

   (d) Consider
$$\begin{aligned}\lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{\cos(x)-\cos(a)}{x-a}\\ &= \lim_{x\to a}\frac{2\sin(\frac{x+a}{2})\sin(\frac{a-x}{2})}{x-a}\\ &= \lim_{x\to a}(-1)\sin(\frac{x+a}{2})\ \frac{\sin(\frac{a-x}{2})}{\frac{a-x}{2}}\\ &= \lim_{x\to a}(-1)\sin(\frac{x+a}{2})\ \lim_{x\to a}\frac{\sin(\frac{a-x}{2})}{\frac{a-x}{2}}\\ &\qquad\text{(provided these limits exist)}\\ &= (-1)\sin(a).1\\ &= -\sin(a).\end{aligned}$$
So f is differentiable at $x=a$ with derivative given by $f'(a)=-\sin(a)$.

2. For $x \neq 0$ the combination theorem and composition of functions tell us that $f(x)$ is continuous. Now for $x = 0$, we have

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (x+1)^2 = 1$$

and

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (\sin(2x) + 1) = 1.$$

Thus $\lim_{x \to 0} f(x) = 1 = f(0)$ which means that f is continuous at $x = 0$ as well.

Similarly, for $x \neq 0$ the combination theorem and composition of functions tell us that $f(x)$ is differentiable. Now for $x = 0$, we have

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{(x+1)^2 - 1}{x - 0} = 2$$

and

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{\sin(2x) + 1 - 1}{x - 0} = 2.$$

Thus $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = 2$ which means that f is differentiable at $x = 0$ as well and we have

$$f'(x) = \begin{cases} 2(x+1) & x \geq 0 \\ 2\cos(2x) & x < 0 \end{cases}$$

Now consider this new function $f'(x)$. Is it continuous? differentiable?

For $x \neq 0$, the combination theorems and composition of function theorems tell us that $f'(x)$ is both continuous and differentiable.

Now for $x = 0$ we have

$$\lim_{x \to 0+} f'(x) = \lim_{x \to 0+} 2(x+1) = 2$$

and

$$\lim_{x \to 0-} f'(x) = \lim_{x \to 0-} 2\cos(2x) = 2.$$

Thus $\lim_{x \to 0} f'(x) = 2 = f'(2)$ which means that f' is continuous at $x = 0$ as well.

But for $x = 0$ we have

$$\lim_{x \to 0+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0+} \frac{2(x+1) - 2}{x - 0} = 2$$

and

$$\lim_{x \to 0-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0-} \frac{2\cos(2x) - 2}{x - 0} = -4\sin(2.0) = 0$$

Thus the derivative on the right is not the same as the derivative on the left and so $f'(x)$ is not differentiable at $x = 0$.

3. We need to show that $\lim_{x\to b}\frac{(f+g)(x)-(f+g)(b)}{x-b}$ exists and is equal to $f'(b)+g'(b)$.

$$\begin{aligned}
\lim_{x\to b}\frac{(f+g)(x)-(f+g)(b)}{x-b} &= \lim_{x\to b}\frac{f(x)+g(x)-f(b)-g(b)}{x-b}\\
&= \lim_{x\to b}\frac{f(x)-f(b)+g(x)-g(b)}{x-b}\\
&= \lim_{x\to b}\frac{f(x)-f(b)}{x-b}+\lim_{x\to b}\frac{g(x)-g(b)}{x-b}\\
&\qquad\text{(provided these limits exists)}\\
&= f'(b)+g'(b).
\end{aligned}$$

4. (a) As the functions e^x, $\cos(x)$ and 1 are differentiable everywhere, using the Combination theorem for differentiation we get that $f(x)=e^x\cos(x)+1$ is differentiable everywhere. Its derivative is given by

$$f'(x)=e^x\cos(x)-e^x\sin(x).$$

(b) As the functions 1 and x are differentiable everywhere and $1+x^4\neq 0$ (for all x), using the Combination theorem for differentiability, we get that $f(x)=\frac{1+x^2}{1+x^4}$ is differentiable everywhere. Its derivative is given by

$$f'(x)=\frac{2x-4x^3-2x^5}{(1+x^4)^2}.$$

(c) As the functions $\sin(x)$ and $\cos(x)$ are differentiable everywhere, using the combination theorem we see that $\tan(x)=\frac{\sin(x)}{\cos(x)}$ is differentiable for all x with $\cos(x)\neq 0$, i.e. for all $x\neq\frac{\pi}{2}+n\pi$. Hence, using the combination theorem again we see that $\tan^3(x)$ is differentiable for all $x\neq\frac{\pi}{2}+n\pi$. Its derivative is given by

$$f'(x)=3\frac{\tan^2(x)}{\cos^2(x)}.$$

(d) For $x\neq 0$ we can use the combination theorem and composition of functions to deduce that f is differentiable. Now when $x=0$ we use the Sandwich rule. First note that for all $x\in\mathbb{R}$ we have

$$-x^2\leq f(x)\leq x^2.$$

Moreover, $-0^2=f(0)=0^2$ and the functions x^2 and $-x^2$ are differentiable at $x=0$. We deduce from the Sandwich rule that $f(x)$ is differentiable at $x=0$ with $f'(0)=0$. Thus the derivative $f'(x)$ of $f(x)$ is given by

$$f'(x)=\begin{cases}2x\cos(1/x)+\sin(1/x) & x\neq 0\\ 0 & x=0\end{cases}$$