

## Real Analysis: Solutions to Exercise Sheet 8

1. (a) Let  $x_1 < x_2 \in [a, b]$ . We want to show that  $f(x_1) > f(x_2)$ . As  $f$  satisfies the hypotheses of the Mean Value Theorem on  $[a, b]$  it also does on  $[x_1, x_2]$ . So applying the MVT to  $f$  on  $[x_1, x_2]$  we can find  $c \in (x_1, x_2)$  such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0$$

by assumption. As  $x_2 - x_1 > 0$  we must have  $f(x_2) - f(x_1) < 0$  and so  $f(x_2) < f(x_1)$  as required.

- (b) Let  $x_1 < x_2 \in [a, b]$ . We want to show that  $f(x_1) = f(x_2)$ . As  $f$  satisfies the hypotheses of the Mean Value Theorem on  $[a, b]$  it also does on  $[x_1, x_2]$ . So applying the MVT to  $f$  on  $[x_1, x_2]$  we can find  $c \in (x_1, x_2)$  such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

by assumption. So we have  $f(x_2) - f(x_1) = 0$  and thus  $f(x_2) = f(x_1)$  as required.

2. Consider the function  $h(x) = f(x) - g(x)$ . Then  $h$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  (as  $f$  and  $g$  are). Moreover we have

$$h'(x) = f'(x) - g'(x) = 0$$

by assumption. Thus applying the result in question 1 (b) we get that  $h(x) = k$  a constant. Thus  $h(x) = f(x) - g(x) = k$  and so  $f(x) = g(x) + k$  as required.

3. We can always find  $x_1$  such that  $p(x_1) = x_1^3 + ax_1 + b > 0$  (take  $x_1$  large enough so that  $x_1^3 + ax_1 > -b$ ). Also we can find  $x_2$  such that  $p(x_2) < 0$  (take  $x_2$  large negative such that  $x_2^3 + ax_2 < -b$ ). So applying the Intermediate Value Theorem, the polynomial  $p(x)$  has a real root. Now we need to show that it has only one root. Suppose for a contradiction that it had two real roots  $a$  and  $b$ . This means that  $p(a) = p(b) = 0$ . So we can apply Rolle's theorem and get  $c \in (a, b)$  with  $p'(c) = 0$ . But  $p'(x) = 3x^2 + a > 0$  (as  $a > 0$ ). This is a contradiction.

4. (a) Consider the function  $f : [a, b] \rightarrow \mathbb{R}$  defined by  $f(x) = \sin(x)$ . Then  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  so we can apply the Mean Value Theorem to get  $c \in (a, b)$  with

$$\frac{\sin(b) - \sin(a)}{b - a} = \cos(c).$$

Now take the absolute value on both sides to get

$$\frac{|\sin(b) - \sin(a)|}{|b - a|} = |\cos(c)| \leq 1.$$

Multiplying both sides of the inequality by  $|b - a|$  we get

$$|\sin(b) - \sin(a)| \leq |b - a|$$

as required.

- (b) Consider the function  $f : [81, 83] \rightarrow \mathbb{R}$  defined by  $f(x) = \sqrt{x}$ . Then  $f$  is continuous on  $[81, 83]$  and differentiable on  $(81, 83)$  so we can apply the Mean Value Theorem to get  $c \in (81, 83)$  with

$$\frac{\sqrt{83} - \sqrt{81}}{83 - 81} = \frac{1}{2} \cdot \frac{1}{\sqrt{c}}.$$

Thus we get

$$\sqrt{83} - 9 = \frac{1}{\sqrt{c}}.$$

As  $81 < c < 83$  we have  $\frac{1}{\sqrt{83}} < \frac{1}{\sqrt{c}} < \frac{1}{9}$ . Also we have  $\frac{1}{\sqrt{100}} < \frac{1}{\sqrt{83}}$ . Thus we get

$$\frac{1}{10} < \sqrt{83} - 9 < \frac{1}{9}.$$

5. (a) False. Take  $f : [0, 2] \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 2 & 1 < x \leq 2 \end{cases}$$

Then there is no  $c \in (0, 2)$  with  $f'(c) = \frac{2-1}{2-0} = \frac{1}{2}$ . (Draw a graph of  $f(x)$  to see that).

- (b) True. Take  $f : [-1, 1] \rightarrow \mathbb{R}$  defined by  $f(x) = |x|$ . Then  $f$  is continuous but there is no  $c \in (-1, 1)$  with  $f'(c) = \frac{1-1}{2} = 0$ .
- (c) False. Take  $f(x) = x^3$ . Then  $f'(0) = 0$  but  $f$  has no maximum or minimum at  $x = 0$ .