Real Analysis: Solutions to Coursework 1

1. First note that the graph of $f(x) = x^2 + x - 6$ is a parabola facing upwards, crossing the x-axis at $x = -3$ and $x = 2$ and having a minimum at $x = -\frac{1}{2}$ with $f(-\frac{1}{2}) = -6.25$.

   (a) False. Take for example $x = -7$ then there is no $y \in \mathbb{R}$ with $y^2 + y - 6 \leq -7$ as we always have $y^2 + y - 6 \geq -6.25$.

   (b) True. Take for example $x = -7$ then for all $y \in \mathbb{R}$ we have $y^2 + y - 6 \geq -6.25 > -7$.

   (c) True. take any x, y with $x, y \geq -\frac{1}{2}$, for instance $x = 0$ and $y = 1$, then
$$x^2 + x - 6 = -6 < y^2 + y - 6 = -4.$$

   (d) False. Take any x, y with $x, y \leq -\frac{1}{2}$, for instance $x = -2$ and $y = -1$, then $x < y$ but
$$x^2 + x - 6 = -4 > y^2 + y - 6 = -6.$$

2. (a) On the one hand we have
$$\begin{array}{rcll} x & < & y & \\ x + x & < & y + x & \text{using (O2)} \\ \frac{1}{2}(x + x) & < & \frac{1}{2}(y + x) & \text{using (O3)} \\ x & < & \frac{x + y}{2}. & \end{array}$$
   On the other hand we have
$$\begin{array}{rcll} x & < & y & \\ x + y & < & y + y & \text{using (O2)} \\ \frac{1}{2}(x + y) & < & \frac{1}{2}(y + y) & \text{using (O3)} \\ \frac{x + y}{2} & < & y. & \end{array}$$

   (b) Suppose, for a contradiction, that the set $[2, 5)$ had a maximum M. Then $M \in [2, 5)$, i.e. $2 \leq M < 5$ and $M \geq x$ for all $x \in [2, 5)$. Now, using (a), we know that
$$2 \leq M < \frac{M + 5}{2} < 5.$$
   Thus $\frac{M+5}{2} \in [2, 5)$ but $\frac{M+5}{2} > M$, this is a contradiction.

3. (a) max=sup=6, min $\nexists$, inf=-7.

   (b) $\{x \in \mathbb{R} : |x + 2| < 3\} = (-5, 1)$. So we have max $\nexists$, min $\nexists$, inf=-5, sup=1.

   (c) max $\nexists$, sup=$\frac{1}{5}$, min $\nexists$, inf=$-\frac{1}{7}$.

(d) max=sup=$\frac{12}{35}$, min $\nexists$, inf=0.

4. (a) Let $\epsilon > 0$. We need to find $N(\epsilon)$ such that $\forall n > N(\epsilon)$ we have $|\frac{7n-2}{2n+3} - \frac{7}{2}| < \epsilon$. Now we have

$$
\begin{aligned}
& |\frac{7n-2}{2n+3} - \frac{7}{2}| < \epsilon \\
\Leftrightarrow \quad & |\frac{-25}{4n+6}| < \epsilon \\
\Leftrightarrow \quad & \frac{25}{4n+6} < \epsilon \\
\Leftrightarrow \quad & 4n+6 > \frac{25}{\epsilon} \\
\Leftrightarrow \quad & n > \frac{25}{4\epsilon} - \frac{3}{2}.
\end{aligned}
$$

So if we take $N(\epsilon)$ to be the smallest integer greater or equal to $\frac{25}{4\epsilon} - \frac{3}{2}$ then tracing the argument backwards we have that for all $n > N(\epsilon)$ we have

$$|\frac{7n-2}{2n+3} - \frac{7}{2}| < \epsilon$$

as required.

(b) Take $\epsilon = \frac{1}{2}$. Then it is impossible to find $N(\frac{1}{2})$ with the property that for all $n > N(\frac{1}{2})$, $|x_n + 1| < \frac{1}{2}$ as whatever choice we make for $N(\frac{1}{2})$ there is always an integer $n > N(\frac{1}{2})$ not divisible by 3 and in this case $x_n = \frac{1}{n^2}$. So we have $|\frac{1}{n^2} + 1| > 1$ and thus $|\frac{1}{n^2} + 1| \not< \frac{1}{2}$.

5. (a) False. Take for instance $(x_n) = (3 - \frac{1}{n})$. Then $x_n < 3$ for all $n \in \mathbb{N}$ but $(x_n) \to 3$ as $n \to \infty$.

(b) False. Take for instance $(x_n) = (0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, \ldots)$. Then $x_n \in [0, 1)$ for all $n \in \mathbb{N}$ but (x_n) is divergent.

(c) True. Take for instance $(x_n) = (\frac{1}{n})$ then $x_n > 0$ for all $n \in \mathbb{N}$ and $x_n \to 0$ as $n \to \infty$.

(d) True. Take for instance $(x_n) = (-\frac{1}{n})$ then $x_{n+1} > x_n$ for all $n \in \mathbb{N}$ (i.e. (x_n) is strictly increasing) and it is convergent as $x_n \to 0$ as $n \to \infty$.