Real Analysis: Solutions to Coursework 2

1.
$$\lim_{x\to -1-} f(x) = \lim_{x\to -1-} x^2+2x-3 = -4$$

and
$$\lim_{x\to -1+} f(x) = \lim_{x\to -1+} x^3 = -1$$

so $\lim_{x\to -1} f(x)$ does not exists (as $\lim_{x\to -1-} f(x) \neq \lim_{x\to -1+} f(x)$).
$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x^3 = 0$$

and
$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \sin(x) = 0$$

so $\lim_{x\to 0} f(x) = 0$.
$$\lim_{x\to \frac{\pi}{2}-} f(x) = \lim_{x\to \frac{\pi}{2}-} \sin(x) = 1$$

and
$$\lim_{x\to \frac{\pi}{2}+} f(x) = \lim_{x\to \frac{\pi}{2}+} \frac{2x}{\pi} = 1$$

so $\lim_{x\to \frac{\pi}{2}} f(x) = 1$.

Now, when $x \neq -1, 0, \frac{\pi}{2}$, f is continuous as $1, x, \sin(x)$ are continuous (using the Combination theorem).

At $x=-1$, f is not continuous as $\lim_{x\to -1} f(x)$ does not exists.

At $x=0$, f is not continuous as $\lim_{x\to 0} f(x) = 0 \neq f(0) = 1$.

At $x=\frac{\pi}{2}$, f is continuous as $\lim_{x\to \frac{\pi}{2}} f(x) = 1 = f(\frac{\pi}{2})$.

2. We show that f is continuous at every point $x=b$. Let $\epsilon > 0$. We need to find $\delta > 0$ such that whenever $|x-b| < \delta$ we have $|(7x-4)-(7b-4)| < \epsilon$.

   Now
$$|(7x-4)-(7b-4)| = |7(x-b)| = 7|x-b|$$

   so if we take $\delta = \frac{\epsilon}{7}$ then whenever $|x-b| < \frac{\epsilon}{7}$ we have
$$|(7x-4)-(7b-4)| = 7|x-b| < 7\frac{\epsilon}{7} = \epsilon$$

   as required.

3. (a) $f(x) = 2\ln(x) + \sqrt{x} - 2$ is continuous on $[1,2]$ and we have $f(1) = -1 < 0$ and $f(2) = 2\ln 2 + \sqrt{2} - 2 > 0$. Applying the Intermediate Value Theorem we can find $1 < x < 2$ satisfying $f(x) = 0$.

(b) The function $p(x)$ is continuous everywhere. Now $p(-1) = -3 < 0$, $p(0) = 1 > 0$ and $p(4) = -4^3 + 1 < 0$. Thus applying the Intermediate Value Theorem twice (on $[-1, 0]$ and on $[0, 4]$) we can find $-1 < x_1 < 0$ and $0 < x_2 < 4$ such that $p(x_1) = p(x_2) = 0$.

4. Consider the function $g : [0, \frac{1}{2}] \to \mathbb{R}$ defined by $g(x) = f(x) - f(x + \frac{1}{2})$. Using the Combination theorem and composition of functions, g is continuous on $[0, \frac{1}{2}]$. Now

$$g(0) = f(0) - f(\frac{1}{2})$$
$$g(\frac{1}{2}) = f(\frac{1}{2}) - f(1) = f(\frac{1}{2}) - f(0) = -g(0)$$

Either $g(0) = g(\frac{1}{2}) = 0$ but then $f(0) = f(\frac{1}{2})$ and we are done (take $c = 0$).

Or one of $g(0)$ and $g(\frac{1}{2})$ is positive and the other is negative. Using the Intermediate Value Theorem, we can find $0 < c < \frac{1}{2}$ with $g(c) = 0$, i.e. $f(c) - f(c + \frac{1}{2}) = 0$, i.e. $f(c) = f(c + \frac{1}{2})$.

We can identify all points on the equator with the points in the interval $[0, 1]$ if we identify 0 and 1. Now the function giving the temperature at each point on the equator can be assumed to be a continuous function f (there is no jump in temperature) defined on $[0, 1]$ with $f(0) = f(1)$. Applying the result above, we can find a point c on the equator between 0 and $\frac{1}{2}$ such that the temperature at c and at $c + \frac{1}{2}$ (which is on the opposite side of the earth) have the same value.

5.

$$\begin{aligned}
\lim_{x \to b} \frac{f(x) - f(b)}{x - b} &= \lim_{x \to b} \frac{(3x^2 - 5x + 7) - (3b^2 - 5b + 7)}{x - b} \\
&= \lim_{x \to b} \frac{3(x^2 - b^2) - 5(x - b)}{x - b} \\
&= \lim_{x \to b} (3(x + b) - 5) = 3(b + b) - 5 \\
&= 6b - 5 = f'(b).
\end{aligned}$$

6. When $x \neq -2$, using the Combination Theorem and composition of functions, $f(x)$ is continuous and differentiable (as $1, x$ and $\sin(x)$ are).

Is $f(x)$ continuous at $x = -2$?

$$\lim_{x \to -2-} f(x) = \lim_{x \to -2-} \sin(x + 2) = 0$$

and

$$\lim_{x \to -2+} f(x) = \lim_{x \to -2+} x^2 + 5x + 6 = 0$$

So $\lim_{x \to -2} f(x) = 0 = f(-2)$ and f is continuous at $x = -2$.

Is f differentiable at $x = -2$?

Consider the derivative on the right:

$$\lim_{x \to -2-} \frac{f(x) - f(-2)}{x + 2} = \lim_{x \to -2-} \frac{\sin(x + 2) - 0}{x + 2} = 1.$$

Consider the derivative on the left:

$$\lim_{x\to -2+}\frac{f(x)-f(-2)}{x+2}=\lim_{x\to -2+}\frac{x^2+5x+6-0}{x+2}=\lim_{x\to -2+}x+3=1.$$

Thus the derivative at $x=-2$ exists and is equal at 1.

Now $f'(x)$ is given by

$$f'(x)=\begin{cases}\cos(x+2) & x<-2\\ 2x+5 & x\geq -2\end{cases}$$

When $x\neq -2$, using the Combination Theorem and Composition of functions we get that $f'(x)$ is continuous and differentiable (as $1, x$ and $\cos(x)$ are).

Is $f'(x)$ continuous at $x=-2$?

$$\lim_{x\to -2-}f'(x)=\lim_{x\to -2-}\cos(x+2)=1$$

and

$$\lim_{x\to -2+}f'(x)=\lim_{x\to -2+}2x+5=1$$

So $\lim_{x\to -2}f'(x)=1=f'(-2)$ and f' is continuous at $x=-2$.

Is f' differentiable at $x=-2$?
Consider the derivative on the right:

$$\lim_{x\to -2-}\frac{f'(x)-f'(-2)}{x+2}=\lim_{x\to -2-}\frac{\cos(x+2)-1}{x+2}=-\sin(0)=0.$$

Consider the derivative on the left:

$$\lim_{x\to -2+}\frac{f'(x)-f'(-2)}{x+2}=\lim_{x\to -2+}\frac{2x+5-1}{x+2}=\lim_{x\to -2+}2=2.$$

Thus f' is not differentiable at $x=-2$.

7. (a) True. Take for example $f(x)=x$ for $x\neq \frac{1}{2}$ and $f(\frac{1}{2})=0$. This function is continuous everywhere except at $x=\frac{1}{2}$.

   (b) False. Take for example $f(x)=\frac{1}{x-2}$. This function is continuous on $(2,5]$ but is not bounded (it has an asymptote at $x=2$).

   (c) True. Take for example $f(x)=-x$ for $x\in[-1,1)$. This function is continuous, bounded on $[-1,1)$ (as $-1<f(x)\leq 1$), but it does not attain a minimum value on $[-1,1)$.

   (d) False. Take for example $f(x)=-x$ for $x\neq 0$ and $f(0)=1$. Then $f([-1,1])\subseteq[-1,1]$ but there is no $x\in[-1,1]$ with $f(x)=x$.

   (e) True. Take for example $f(x)=|x-1|$. Then f is continuous on $[0,2]$ but it is not differentable at $x=1$.