Real Analysis: Solutions

1. (a) i. The set S is bounded above if there exists $H \in \mathbb{R}$ such that $x \leq H$ for all $x \in S$.
   ii. The real number H is called an upper bound for S if $x \leq H$ for all $x \in S$.
   iii. The real number α is the supremum of S if α is the smallest upper bound for S.
   iv. The real number β is the maximum of S if $\beta \in S$ and $x \leq \beta$ for all $x \in S$.

[2]

(b) i. False. Take for example $S = [0, 1)$ then S is bounded above (by 1) but it has no maximum.
   ii. True. Let β be the maximum of S then we claim that β is also the supremum. As $x \leq \beta$ for all $x \in S$, we have that β is an upper bound. Moreover it is the smallest upper bound as $\beta \in S$ and so any $H < \beta$ cannot be an upper bound for S.
   iii. True. Take $S = [0, 1)$ then 1 is the supremum of S but S has no maximum.

[6]

(c) i. This set is equal to $[-8, -4]$. min=inf=-8, max=sup=-4.
   ii. inf=$-\frac{1}{5}$, sup=$\frac{1}{3}$, no max, no min.
   iii. inf=0, no min, max=sup=$\frac{8}{15}$.
   iv. Factorising $5x^2 + 9x - 2 = (5x - 1)(x + 2)$ we see that this set is equal to $(-2, \frac{1}{5})$. no min, no max, inf=-2, sup=$\frac{1}{5}$.

[8]

(d) Suppose, for a contradiction that $[1, 5)$ had a maximum, call it M. As $M \in [1, 5)$ we have $1 \leq M < 5$. Consider the number $\frac{M+5}{2}$. We have

$$1 \leq M < \frac{M+5}{2} < 5.$$

So $\frac{M+5}{2} \in [1, 5)$ and $M < \frac{M+5}{2}$ this contradicts the maximality of M. Thus $[1, 5)$ has no maximum.

[4]

2. (a) For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n > N$ we have $|x_n - l| < \epsilon$.
   i. Fix $\epsilon > 0$. We want to find N such that $\forall n > N$ we have

$$|\frac{6n-5}{2n+3} - 3| = \frac{14}{2n+3} < \epsilon.$$

This is equivalent to finding N such that $\forall n > N$ we have

$$2n > \frac{14}{\epsilon} - 3$$

or

$$n > \frac{7}{\epsilon} - \frac{3}{2}.$$

Take N to be the smallest (or in fact any) integer greater or equal to $\frac{7}{\epsilon} - \frac{3}{2}$. Then for all $n > N$ we have

$$n > N \geq \frac{7}{\epsilon} - \frac{3}{2}$$

and so

$$\left|\frac{6n-5}{2n+3} - 3\right| < \epsilon$$

as required.

ii. Take $\epsilon = \frac{1}{6}$, then we cannot find an appropriate N. Indeed, whatever N might be, we can always find a multiple of 3, n say, which is greater than N and in this case we have

$$|x_n - 0| = \frac{1}{3} > \frac{1}{6}.$$

[8]

(b) i. We prove this by induction on n. For $n = 1$ the result is clear as $2 < x_1 = 3 < 5$. Now suppose the result holds for n and prove it for $n+1$. We have

$$14 < 7x_{n+1} = x_n^2 + 10 < 35$$

and so

$$2 < x_{n+1} < 5$$

as required.

ii.

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{7}(x_n^2 + 10) - x_n \\ &= \frac{1}{7}(x_n^2 - 7x_n + 10) \\ &= \frac{1}{7}(x_n - 2)(x_n - 5) < 0 \end{aligned}$$

using the first part.

iii. As (x_n) is decreasing and bounded below it is convergent. In order to find its limit, l say, we take the limit as n tends to infinity on both sides of the equation

$$7x_{n+1} = x^2 + 10.$$

Using the Combination theorem for limits we get

$$7l = l^2 + 10$$

and so $l^2 - 7l + 10 = 0$ and $l = 2$ or 5. As (x_n) is decreasing and $2 < x_n < 5$ we must have $l = 2$.

[6]

(c) i. False. Take $(x_n) = (\frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{1}{4}, ...)$ then (x_n) is not convergent.

ii. True. Take $(x_n) = (\frac{1}{n})$.

iii. True. Take $(x_n) = ((-1)^n n)$.

[6]

3. (a) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that $f(a) = \alpha$ and $f(b) = \beta$ then for every γ between α and β there is $a < c < b$ satisfying $f(c) = \gamma$.

[2]

(b) Consider the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - x$. Then g is continuous on $[a, b]$. Moreover we have

$$g(a) = f(a) - a \geq 0,$$
$$g(b) = f(b) - b \leq 0.$$

If $f(a) = a$ or $f(b) = b$ then we are done. So assume that $f(a) \neq a$ and $f(b) \neq b$ then we have

$$g(a) = f(a) - a > 0,$$
$$g(b) = f(b) - b < 0.$$

Using the intermediate value theorem we can find $a < c < b$ with $g(c) = 0$. This means that $f(c) - c = 0$ and so $f(c) = c$ as required.

[6]

(c) i. Consider the function $f : [0, \frac{\pi}{2}] \to [0, \frac{\pi}{2}]$ defined by $f(x) = \cos(x)$. Then f is continuous so applying (b) we know that f has a fixed point in the interval $[0, \frac{\pi}{2}]$.

ii. First note that $p(x)$ is continuous everywhere. Now we have $p(0) = -9 < 0$, $p(2) = 23 > 0$ and $p(-3) = 18 > 0$. Thus applying the intermediate value theorem twice we see that $p(x)$ has one root in the interval $(-3, 0)$ and one root in the interval $(0, 2)$.

iii. Consider the function $f(x) = 2\tan(x) - 1 - \cos(x)$ then $f(x)$ is continuous on $[0, \frac{\pi}{4}]$. Now $f(0) = -2$ and $f(\frac{\pi}{4}) = 1 - \frac{1}{\sqrt{2}} > 0$. Thus applying the intermediate value theorem we see that $f(x) = 0$ for some $x \in (0, \frac{\pi}{4})$.

[6]

(d) i. True. Take $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 1 & x \neq 1 \\ 0 & x = 1 \end{cases}$$

ii. False. Take $f : (0,1) \to (0,1)$ defined by $f(x) = x^2$. Then f is continuous but there is no x with $x^2 = x$ (as 0 and 1 do not belong to the domain).

iii. False. Take $f : (0,1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. Then f is continuous but it does not attain a maximum value (as it is not bounded).

[6]

4. (a) i. f is continuous at point $b \in (a,c)$ if and only if $\lim_{x\to b} f(x)$ exists and is equal to $f(b)$.

ii. f is differentiable at point $b \in (a,c)$ if and only if

$$\lim_{x\to b} \frac{f(x)-f(b)}{x-b} \qquad \text{exists.}$$

In this case the value of this limit is denoted by $f'(b)$.

iii. Consider the function $F_b : (a,c) \to \mathbb{R}$ defined by

$$F_b(x) = \begin{cases} \frac{f(x)-f(b)}{x-b} & x \neq b \\ f'(b) & x = b \end{cases}$$

Then F_b is continuous at point b as $\lim_{x\to b} \frac{f(x)-f(b)}{x-b}$ exists and is equal to $f'(b)$. Now $f(x) = F_b(x)(x-b) + f(b)$ for all x. Thus f is continuous at b (using Combination theorem).

iv. Take $f(x) = |x|$. Then f is continuous everywhere but it is not differentiable at $x = 0$.

[10]

(b) As $x+1$ and e^x are continuous for all x, we have that $f(x)$ is continuous for $x \neq 0$. Now we have

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (x+1) = 1$$

and

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} e^x = 1$$

so $\lim_{x\to 0} f(x)$ exists and is equal to $1 = f(0)$. Thus f is continuous at $x = 0$.
As $x+1$ and e^x are differentiable for all x, we have that $f(x)$ is differentiable for $x \neq 0$. Now we have

$$\lim_{x\to 0-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0-} \frac{x}{x} = 1$$

and

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0+} \frac{e^x-1}{x} = 1$$

so $\lim_{x\to 0} \frac{f(x)-f(0)}{x}$ exists and is equal to 1. Thus f is differentiable at $x = 0$. The derivative $f'(x)$ of $f(x)$ is given by

$$f'(x) = \begin{cases} 1 & x \leq 0 \\ e^x & x > 0 \end{cases}$$

For $x \neq 0$, $f'(x)$ is continuous (as 1 and e^x are continuous). For $x = 0$, let us consider the limit of $f'(x)$ as $x \to 0$.

$$\lim_{x\to 0+} f'(x) = \lim_{x\to 0+} 1 = 1,$$

$$\lim_{x\to 0-} f'(x) = \lim_{x\to 0-} e^x = 1.$$

Thus $\lim_{x\to 0} f'(x)$ exists and is equal to $1 = f'(0)$. Hence $f'(x)$ is continuous everywhere.

For $x \neq 0$, $f'(x)$ is differentiable (as 1 and e^x are differentiable). For $x = 0$, let us consider

$$\lim_{x\to 0+} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0+} \frac{1-1}{x} = 0,$$

$$\lim_{x\to 0-} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0-} \frac{e^x-1}{x} = 1.$$

Thus $\lim_{x\to 0} \frac{f'(x)-f'(0)}{x-0}$ doesn't exists and $f'(x)$ is not differentiable at $x = 0$.

[10]

5. (a) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $a < c < b$ with

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

[2]

(b) Let $a \leq x_1 < x_2 \leq b$. We want to show that $f(x_1) < f(x_2)$. Apply the Mean Value Theorem to $f : [x_1, x_2] \to \mathbb{R}$ to get $x_1 < c < x_2$ with

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

By assumption we have $f'(c) > 0$ and so $\frac{f(x_2)-f(x_1)}{x_2-x_1} > 0$. Now as $x_2 > x_1$ we must have $f(x_2) > f(x_1)$ as required.

[6]

(c) i. Using the MVT for $f : [49, 51] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ we can find $49 < c < 51$ such that

$$f'(c) = \frac{1}{2}.\frac{1}{\sqrt{c}} = \frac{f(51)-f(49)}{51-49} = \frac{\sqrt{51}-7}{2}.$$

As $49 < c < 51$ we have

$$\frac{1}{2}.\frac{1}{\sqrt{51}} < \frac{\sqrt{51}-7}{2} < \frac{1}{2}.\frac{1}{7}$$

and hence (as $\frac{1}{8} < \frac{1}{\sqrt{51}}$) we have

$$\frac{1}{8} < \sqrt{51}-7 < \frac{1}{7}$$

as required.

ii. Using the MVT for $f : [a,b] \to \mathbb{R}$ given by $f(x) = \ln(x)$ we can find $a < c < b$ such that

$$f'(c) = \frac{1}{c} = \frac{\ln(b)-\ln(a)}{b-a}.$$

So we get

$$\frac{1}{b} < \frac{\ln(b)-\ln(a)}{b-a} < \frac{1}{a}.$$

And hence

$$1-\frac{a}{b} < \ln(\frac{b}{a}) < \frac{b}{a}-1$$

as required.

iii. Using the MVT for $f : [a,b] \to \mathbb{R}$ given by $f(x) = \sin^{-1}(x)$ we can find $0 < a < c < b < 1$ such that

$$f'(c) = \frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}(b)-\sin^{-1}(a)}{b-a}.$$

So we get

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}(b)-\sin^{-1}(a) < \frac{b-a}{\sqrt{1-b^2}}$$

as required.

[12]