CITY UNIVERSITY LONDON

Entanglement Entropy of Quantum Spin Chains:

Infinitely Degenerate Ground States

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Overview of the talk

- Entanglement entropy

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- Cyclic permutation operators and twist fields

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This talk is mainly based on the following publications:

O.A. Castro-Alvaredo and B. Doyon, arXiv:1011.4706
arXiv:1103.3247

Bi-partite Entanglement Entropy

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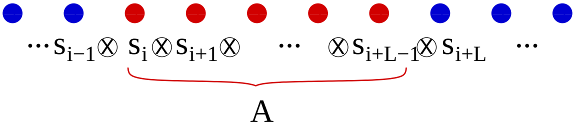

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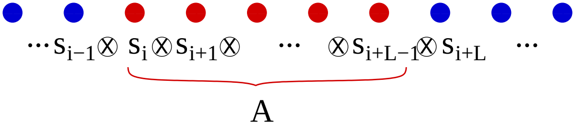

then we define

Von Neumann Entanglement Entropy

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\bar{A}}(|\Psi\rangle\langle\Psi|)$$

and $|\Psi\rangle$ a ground state.

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- Other entropies may also be defined such as Rényi's

Other Entropies

$$S_A^{(n)} = \frac{\log(\mathrm{Tr}_A(\rho_A^n))}{1-n}$$

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Replica Trick

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1}\frac{d}{dn}\mathrm{Tr}_A(\rho_A^n)$$

- For QFTs it naturally leads to the notion of replica theories on multi-sheeted Riemann surfaces and twist fields (see B. Doyon's talk).

Twist Fields

- In QFT we have seen that the entropy may be expressed in terms of a two-point function of twist fields

$$\boxed{\mathrm{Tr}_A(\rho_A^n) \propto \langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle}$$

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- Recall that the twist field's "mission" is to cyclicly permute the n-copies of a QFT in its replica version.
- We may consider a replica spin chain theory, consisting of n non-interacting copies of some known model and try to construct a local operator which cyclicly permutes the spin of a certain site amongst the various copies.

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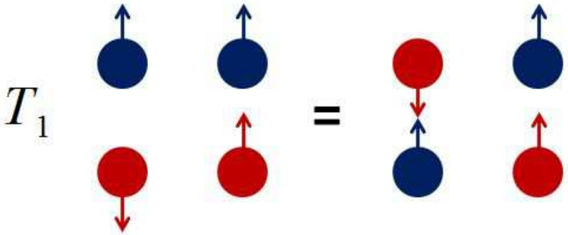

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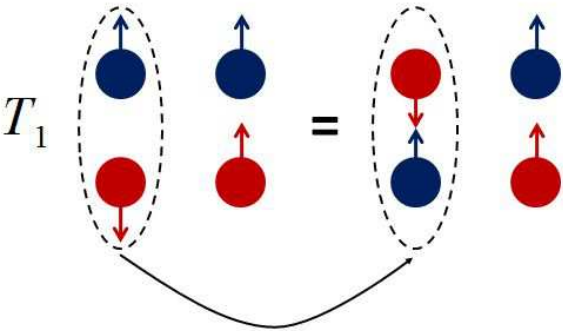

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- The simplest example is the case $n = 2$ and $N = 2$ (two sites and two copies). Let us represent graphically the action of $\mathcal{T}_1$:
- In this simple case, the operator $\mathcal{T}_1$ is nothing but the permutation operator. It may be written as

Cyclic Permutation Operator for n=N=2

$$\mathcal{T}_i = E_{1,i}^{11}E_{2,i}^{11} + E_{1,i}^{12}E_{2,i}^{21} + E_{1,i}^{21}E_{2,i}^{12} + E_{1,i}^{22}E_{2,i}^{22} \quad \text{with} \quad i = 1, 2,$$

where

$$(E_{\alpha,i}^{\epsilon\epsilon'})_{jk} = \delta_{\epsilon,j}\delta_{\epsilon',k} \quad \alpha = 1, \ldots, n \quad \text{and} \quad i = 1, \ldots, N$$

General Formula I

- It is now easy to imagine how several operators $\mathcal{T}_i$ will act for general values of n and N. Graphically let us think of the following product of states:

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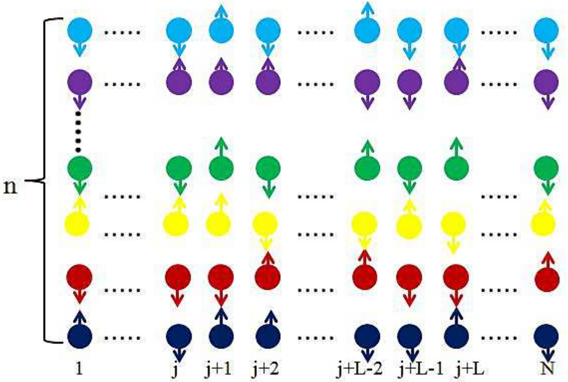

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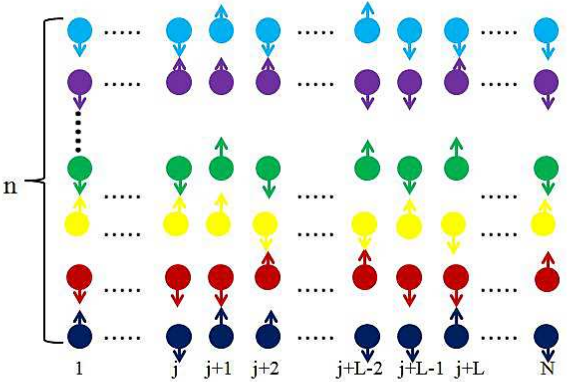
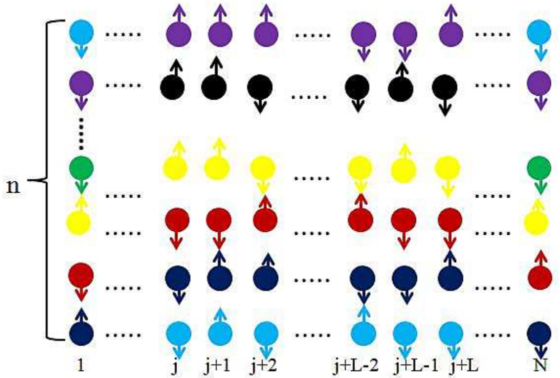


- Acting with the operator

$$\mathcal{T}_j \mathcal{T}_{j+1} \cdots \mathcal{T}_{j+L-1} \mathcal{T}_{j+L}$$

General Formula II

- We obtain a new state



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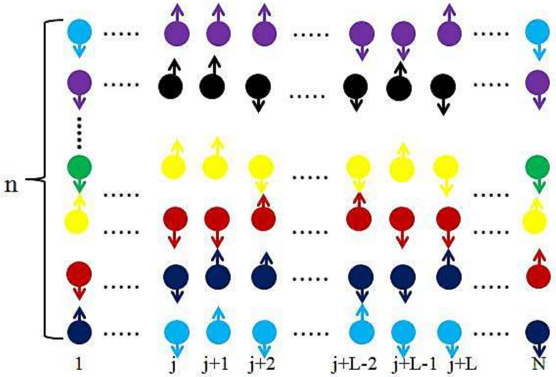

General Cyclic Permutation Operator

$$\mathcal{T}_i = \mathrm{Tr}_{\mathrm{aux}} \left(\prod_{\alpha=1}^{n} T_{\alpha,i;\mathrm{aux}} \right) \quad \text{with} \quad T_{\alpha,i;\mathrm{aux}} = \begin{pmatrix} E^{11}_{\alpha,i} & E^{21}_{\alpha,i} \\ E^{12}_{\alpha,i} & E^{22}_{\alpha,i} \end{pmatrix}_{\mathrm{aux}}$$

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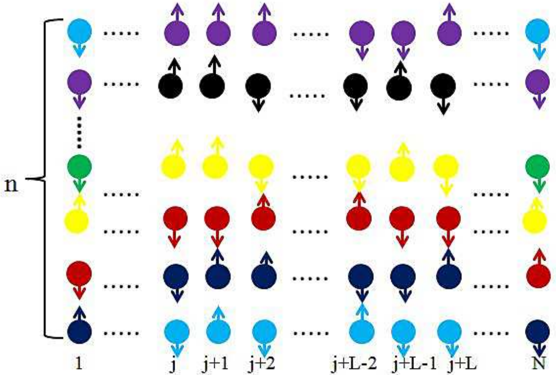

- In the scaling limit, when both $N, L \to \infty$ expectation values of products of cyclic permutation operators play the role of the two point function of twist fields in QFT.

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Entanglement Entropy of Two Spins

$$\begin{aligned} S_{\{1,m+1\}} \quad = \quad & -\frac{(1-z(m)+4s(m))}{4}\log\left[\frac{(1-z(m)+4s(m))}{4}\right] \\ & -\frac{(1-z(m)-4s(m))}{4}\log\left[\frac{(1-z(m)-4s(m))}{4}\right] \\ & -\frac{(1+z(m))}{2}\log\left[\frac{1+z(m)}{4}\right]. \end{aligned}$$

with $\langle \sigma_1^z \sigma_{m+1}^z \rangle = z(m)$ and $\langle \sigma_1^+ \sigma_{m+1}^- \rangle = s(m)$.

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- Computations become very involved whenever more than two spins are considered (e.g. very complex combinatorial problem).
- We therefore decided to look at a theory where correlation functions are so simple, that one may hope to find a general formula for the entropy.
- We obtained those correlation functions simply by taking the $\Delta \to -1^+$ limit of the correlation functions of the XXZ chain. This leads to

The Correlation Functions

$$\lim_{\Delta \to -1^+} \langle E_{j_1}^{\epsilon_1 \epsilon_1'} E_{j_2}^{\epsilon_2 \epsilon_2'} \cdots E_{j_m}^{\epsilon_m \epsilon_m'} \rangle = \frac{1}{2^m} \prod_{j \in B} (-1)^j$$

where B is the subset of sites at which either an operator E^{12} or an operator E^{21} sit.

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Rényi Entropy

$$S_m^{(n)} = -\frac{nm \log 2}{1-n} + \frac{1}{1-n} \log \left(\sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right)^n \right)$$

where m is the number of spins whose entropy is being computed (not necessarily consecutive!).

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Asymptotics

$$S_m = \frac{1}{2}\log\left(\frac{\pi m}{2}\right) + \frac{1}{2} + O\left(m^{-1}\right)$$

$$S_m^{(n)} = \frac{1}{2}\log\left(\frac{\pi m}{2}\right) + \frac{\log(n)}{2(n-1)} + O\left(m^{-1}\right)$$

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- What does this asyptotic behaviour actually mean?

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- The meaning of this “dimension” is relatively simple to explain for the spin-$\frac{1}{2}$ XXX chain.
- There are infinitely many degenerate ground states for this model.
- An infinite subset of these states are factorizable (zero-entropy states) corresponding to choosing all spins to point in a particular direction $\vec{v}$.

Ground states of the spin-$\frac{1}{2}$ XXX spin chain

- Every factorizable ground state $|\Psi\rangle_{\vec{v}}$ may be labeled by a unit vector $\vec{v}$, which takes any possible direction inside the two-dimensional sphere S^2 and such that $\vec{\sigma} \cdot \vec{v}|\Psi\rangle_{\vec{v}} = |\Psi\rangle_{\vec{v}}$.

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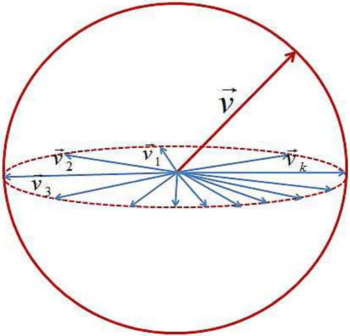
- The particular ground state whose entropies we have computed corresponds to an infinite linear combination where all coefficients c_α are equal to each other and where the vectors $\vec{v}_\alpha$ are such as to generate a great circle on the unit sphere. Graphically...

Different geometries



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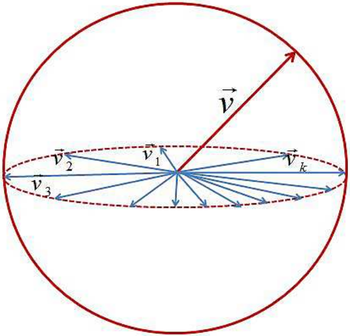

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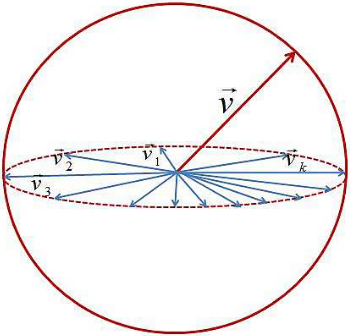

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- More complex linear combinations of basic states give rise to ground states whose entropy might have asymptotics characterized by $d \neq 1$ and even d non integer (fractal sets).

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- The approach is based on the use of replica cyclic permutation operators and their correlation functions.
- Employing this approach we have identify a particular type of universal behaviour of the entanglement entropy which may be found in theories with infinitely degenerate ground state spanned by a basis of zero-entropy states.
- This universal behaviour is characterized by the geometric structure of the support of these states, which may be a fractal one.