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On the solution of the inverse scattering problem for XXZ quantum spin chains

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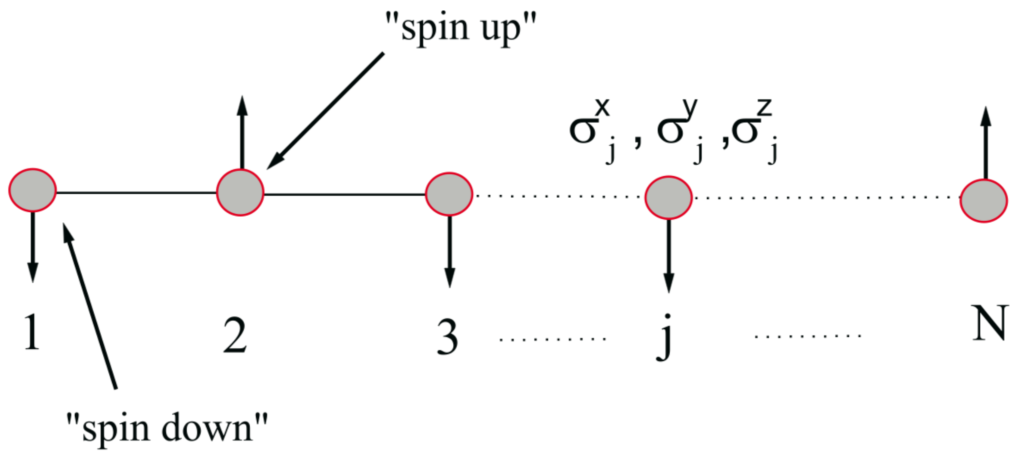
I. CORRELATION FUNCTIONS OF QUANTUM SPIN CHAINS

- Everything that can be measured is related to correlation functions. The knowledge of all correlation functions is the solution of any physical theory.
- Here we will be interested in a particular kind of physical theories: integrable quantum spin chains.



- The figure shows the simplest situation in which the spin at all sites of the chain is $1/2$. The z-component on the spin admits two possible projections ("spin up/spin down") and the physical observables are the Pauli matrices at each site of the chain.

- Amongst the integrable quantum spin chains, the XXX and XXZ spin 1/2 Heisemberg spin chains are probably the most widely studied examples: (W. Heisenberg '28, H. Bethe '31)

- The Hamiltonian describes spin chains with nearest-neighbour interactions:

$$H = \sum_{m=1}^{N} \Big\{ \underbrace{\sigma_m^x \sigma_{m+1}^x}_{X} + \underbrace{\sigma_m^y \sigma_{m+1}^y}_{X} + \underbrace{\Delta \big(\sigma_m^z \sigma_{m+1}^z - 1 \big)}_{Z} \Big\}$$

$\Delta \equiv$ anisotropy parameter ($\Delta = 1$ corresponds to the XXX chain)

- A lot of effort has been made over the last years to compute the correlation functions of these quantum spin chains in a systematic way.

- Two main approaches have been especially successful in this task. The first approach was initiated by M. Jimbo and T. Miwa '95 and the second by N. Kitanine, J.-M. Maillet and V. Terras '99. Conceptually, they are very different approaches. Here I will concentrate on the second one.

II. COMPUTING CORRELATION FUNCTIONS

- A formal definition of two-point functions (which are particular examples of correlation functions) is presented below. Define:

▷ S_j^α: the α-component ($\alpha = x, y, z$) of the spin operator at site j of the chain.

▷ S_k^β: the β-component of the spin operator at site k of the chain.

$$\langle S_j^\alpha S_k^\beta \rangle = \frac{\mathrm{tr}_{\mathcal{H}}\left(S_j^\alpha S_k^\beta e^{-H/kT}\right)}{\mathrm{tr}_{\mathcal{H}}\left(e^{-H/kT}\right)} \underset{T \to 0}{=} \frac{\langle \Psi_g | S_j^\alpha S_k^\beta | \Psi_g \rangle}{\langle \Psi_g | \Psi_g \rangle}$$

- In general, we can attempt to compute correlation functions by finding a consistent mathematical description of the physical states (in particular, the ground state) and of the local operators on the chain.

III. THE QUANTUM STATES: ALGEBRAIC BETHE ANSATZ

- The physical states or eigenstates of the Hamiltonian can be constructed by means of the algebraic Bethe ansatz technique (L.D Faddeev, E.K. Sklyanin and L.A. Takhtajan '79): starting with an R-matrix (a solution of the Yang-Baxter equations) a monodromy matrix can be defined as:

$$T^{(\frac{1}{2})}_{0;1\ldots N}(\lambda;\{\xi\}) = R^{(\frac{1}{2},s_N)}_{0N}(\lambda-\xi_N)\cdots R^{(\frac{1}{2},s_1)}_{01}(\lambda-\xi_1)$$

$$= \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}; \quad t^{(1/2)}(\lambda,\{\xi\}) = (A+D)(\lambda)$$

- where $R^{(\frac{1}{2},s_i)}_{0i}(\lambda) \in \mathbb{C}^2 \otimes \mathbb{C}^{2s_i+1}$ and $\xi_1,\ldots,\xi_N$ are the inhomogeneity parameters (it is useful to introduce them for correlation function computations).

• The Hamiltonian of the chain and the transfer matrix $(A+D)(\lambda)$ have common eigenstates:

$$|\Psi(\{\lambda\})\rangle = B(\lambda_1)\cdots B(\lambda_\ell)|0\rangle \quad \text{with}$$

$$\prod_{k=1}^{\ell} \frac{b(\lambda_j-\lambda_k)}{b(\lambda_k-\lambda_j)} = -d(\lambda_j) \quad \text{Bethe ansatz equations}$$

• Here $|0\rangle$ denotes the completely ferromagnetic reference state (all spins up) and $d(\lambda)$ is the eigenvalue of $D(\lambda)$ on that state

$$D(\lambda)|0\rangle = d(\lambda)|0\rangle = \prod_{j=1}^{N}\left[\prod_{k=1}^{2s_j} b(\lambda-\xi_j-(k-s_j-1/2)\eta)\right]|0\rangle$$

$$A(\lambda)|0\rangle = |0\rangle$$

• For XXZ quantum spin chains $b(\lambda)=\sinh\lambda/\sinh(\lambda+\eta)$ and η is related to the anisotropy parameter in the Hamiltonian $\Delta = 2\cosh\eta$.

IV. THE OPERATORS: THE INVERSE SCATTERING PROBLEM

- Since the quantum states are given in terms of the operators $\{A, B, C, D\}$ it is natural to look for realizations of the operators in terms of the same objects.

- This problem is known as inverse scattering problem and was first solved in J.-M. Maillet, V. Terras and N. Kitanine '99; J.-M. Maillet and V. Terras 2000. They proved that for any spin operator S_j^α and any spin representation s_j:

$$S_j^\alpha = \left[\prod_{k=1}^{j-1} t^{(s_k)}(\xi_k)\right] \Lambda_\alpha^{(s_j)}(\xi_j) \left[\prod_{k=1}^{j} t^{(s_k)}(\xi_k)^{-1}\right] \qquad \alpha = \pm, z$$

$$\Lambda_\alpha^{(s)}(u) := \mathrm{Tr}_0\left[S_0^\alpha T_{0;1\ldots N}^{(s)}(u)\right], \qquad t^{(s)}(u) := \mathrm{Tr}_0\left[T_{0;1\ldots N}^{(s)}(u)\right]$$

- The proof of this expression requires only that the R-matrix, $R_{12}^{(s,s)}(0) = \mathbb{P}_{12}$, where $\mathbb{P}$ is the permutation matrix. Therefore, it holds for any XXX, XXZ and XYZ quantum spin chains.

V. DEALING WITH GENERIC (HIGHER) SPINS: FUSION

- The formula we just saw holds for many kinds of theories, however it is also purely formal if we do not find a mechanism to obtain the traces:

$$\Lambda_\alpha^{(s)}(u) := \mathrm{Tr}_0 \left[S_0^\alpha T_{0;1\ldots N}^{(s)}(u) \right], \qquad t^{(s)}(u) := \mathrm{Tr}_0 \left[T_{0;1\ldots N}^{(s)}(u) \right]$$

- The mechanism needed to construct higher spin objects out of spin 1/2 objects is known fusion and was developed in P.P. Kulish, N. Yu. Reshetikhin and E.K. Sklyanin '81 for XXX chains and in A.N. Kirillov and N. Yu. Reshetikhin '87 for XXZ spin chains.

- For the monodromy matrices the following fusion identities hold:

$$P_{0\hat{0}}T_0^{(\frac{1}{2})}(x^- + s\eta)T_{\hat{0}}^{(s-\frac{1}{2})}(x^-)P_{0\hat{0}}$$
$$= \begin{pmatrix} T_{\langle 0\hat{0}\rangle}^{(s)}(x) & 0 \\ * & \chi(x+(s-1)\eta)T_{(0\hat{0})}^{(s-1)}(x-\eta) \end{pmatrix}$$

- $P_{0\hat{0}} = P_{\langle 0\hat{0}\rangle}^{+} \oplus P_{(0\hat{0})}^{-}$ is a direct sum of projectors on the vector spaces associated to spins s and $s-1$, respectively.

- The operator $\chi(\lambda)$ is the quantum determinant:

$$\chi(\lambda) = A(\lambda^+)D(\lambda^-) - B(\lambda^+)C(\lambda^-)$$

and the variables

$$\lambda^{\pm} = \lambda \pm \eta/2.$$

- These formulae hold both for XXX ($\eta = -i$) and XXZ chains ($q = e^{\eta}$).

- Taking traces on the r.h.s. and l.h.s. of the fusion relation above, recursive relations for the transfer matrices can be obtained:

$$t^{(s)}(x^+) = t^{(\frac{1}{2})}(x + s\eta)t^{(s-\frac{1}{2})}(x) - \chi(x^- + s\eta)t^{(s-1)}(x^-)$$

- and from them (by acting on a quantum state) similar relations are obtained for the eigenvalues of these transfer matrices.

- Now we just need to compute traces of the form $\mathrm{Tr}_0(S_0^\alpha T_0^{(s)}(\lambda))$. For the XXX-case this computation has been performed for arbitrary s (J.-M. Maillet and V. Terras 2000). The main input to do so is to notice that for the XXX case:

$$S_0^\alpha \otimes 1_{\hat{0}} + 1_0 \otimes S_{\hat{0}}^\alpha = S_{\langle 0\hat{0}\rangle}^\alpha \oplus S_{(0\hat{0})}^\alpha$$

- This is the same as saying that the co-product is trivial in this case!

● Multiplying this identity by the fusion relation and taking thereafter the trace they obtained:

$$\Lambda_\alpha^{(s)}(u) = \Lambda_\alpha^{(1/2)}(u^- + s\eta)t^{(s-1/2)}(u^-)$$
$$+t^{(1/2)}(u^- + s\eta)\Lambda_\alpha^{(s-1/2)}(u^-) - \chi(u + (s-1)\eta)\Lambda_\alpha^{(s-1)}(u-\eta).$$

● This recursive relation is solved by:

$$\Lambda_\alpha^{(s)}(u) = \sum_{k=1}^{2s}\left[t^{(s-\frac{k}{2})}\left(u + \frac{k\eta}{2}\right)\Lambda_\alpha^{(\frac{1}{2})}(u^- + (k-s)\eta)\right.$$
$$\left.\times t^{(\frac{k-1}{2})}\left(u^- + \frac{(k-2s)\eta}{2}\right)\right],$$

● Employing this formula we were able to find closed expressions for all form factors of higher spin XXX quantum spin chains.

VI. WHAT CHANGES IN THE XXZ CASE?

- The fundamental difference is that now the co-product is not trivial, in other words there is a deformation parameter so that:

$$S_0^\alpha \otimes D_{\hat{0}}^\alpha(\eta) + D_0^\alpha(\eta) \otimes S_{\hat{0}}^\alpha = S_{\langle 0\hat{0}\rangle}^\alpha \oplus S_{(0\hat{0})}^\alpha$$

- The form of the co-product is slightly non-standard due to our choice of the R-matrix (it is related to the quantum group R-matrix by a twist). For the S^z generators $D^z = 1$, as expected. For the other generators $D^\pm$ are non-trivial. For example for spin 1:

$$\begin{aligned}
&\sigma_0^+ \otimes \mathcal{D}_{\hat{0}}^+(\eta) + \mathcal{D}_0^+(\eta) \otimes \sigma_{\hat{0}}^+, \\
&\sigma_0^- \otimes \mathcal{D}_{\hat{0}}^-(\eta) + \mathcal{D}_0^-(\eta) \otimes \sigma_{\hat{0}}^-, \\
&\sigma_0^z \otimes 1_{\hat{0}} + 1_0 \otimes \sigma_{\hat{0}}^z,
\end{aligned}$$

- with $D^+(\eta) = \text{diag}(1, \cosh\eta)$ and $D^-(\eta) = \text{diag}(\cosh\eta, 1)$. σ^α are the Pauli matrices.

- As a consequence the traces $\mathrm{Tr}_0(S_0^{\pm} T_0^{(s)}(\lambda))$ satisfy slightly different recursive relations (they are the same as before for S^z):

$$\Lambda_{\pm}^{(s)}(u) = \Lambda_{\pm}^{(1/2)}(u^- + s\eta) t_{\pm}^{(s-1/2)}(u^-)$$
$$+ t_{\pm}^{(1/2)}(u^- + s\eta) \Lambda_{\pm}^{(s-1/2)}(u^-) - \kappa \chi(u + (s-1)\eta) \Lambda_{\pm}^{(s-1)}(u - \eta).$$

- Here $\kappa = \cosh \eta$ and $t_{\pm}^{(s)}(\lambda)$ are “deformed” transfer matrices:

$$t_{\pm}^{(s)}(\lambda) = \mathrm{Tr}_0(D_0^{\pm}(\eta) T_0^{(s)}(\lambda))$$

- They also satisfy recursive relations of the same type as $t^{(s)}(\lambda)$:

$$t_{\pm}^{(s)}(u) = t_{\pm}^{(1/2)}(u^- + s\eta) t_{\pm}^{(s-1/2)}(u^-) - \kappa \chi(u + (s-1)\eta) t_{\pm}^{(s-1)}(u - \eta),$$

VII. THE SOLUTION

- The recursive relations above can be solve in a similar way as for the XXX case an yield:

$$\Lambda_{\pm}^{(s)}(u) = \sum_{k=1}^{2s} t_{\pm}^{(\frac{2s-k}{2})}(u+\frac{k\eta}{2})\Lambda_{\pm}^{(\frac{1}{2})}(u^{-}+(k-s)\eta)t_{\pm}^{(\frac{k-1}{2})}(u^{-}+\frac{k-2s}{2}\eta)$$

- The quantum group generators of the XXZ spin chain in the spin s_j representation can be reconstructed as:

$$S_j^{\pm} = \left[\prod_{k=1}^{j-1} t^{(s_k)}(\xi_k)\right] \Lambda_{\pm}^{(s_j)}(\xi_j) \left[\prod_{k=1}^{j} t^{(s_k)}(\xi_k)^{-1}\right]$$

- Problem: $t_{\pm}^{(s)}(\lambda)$ do not act diagonally on the quantum states. The computation of correlation functions and form factors becomes much harder!

VIII. CONCLUSIONS AND OUTLOOK

- In this work we have reconstructed the quantum group generators associated to the XXZ quantum spin chain in terms of the entries of the monodromy matrix A, B, C, D.

- This reconstruction is absolutely generic, it holds for any spin representation at any site of the chain.

- The solution involves a hierarchy of new objects $t_{\pm}^{(s)}(\lambda)$, which we have called "deformed" transfer matrices. Contrarily to the transfer matrices, these new operators **do not** act diagonally on the quantum states of the chain.

- This property complicates the computation of form factors and correlation functions for higher spins. This can be seen already for spin 1, a case for which we computed all form factors.

- This means that in order to be able to exploit these formulae successfully, a detailed study of the action of the operators $t_{\pm}^{(s)}(\lambda)$ on generic quantum states should be carried out.