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Branch-point twist fields and entanglement entropy in integrable quantum field theory

Olalla Castro Alvaredo

Centre for Mathematical Science

City University London

in collaboration with

John Cardy (Oxford University and All Souls College)

Benjamin Doyon (Oxford University)

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WHAT THIS WORK IS ABOUT

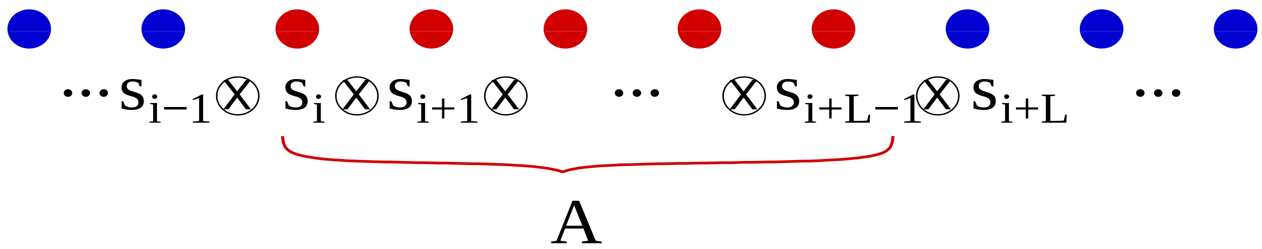
- Quantity we want to compute $\Rightarrow$ the **entanglement entropy** of quantum systems. More precisely we want to consider integrable QFTs and to compute the leading correction to the large-distance behaviour of the entropy
- Methods we want to employ $\Rightarrow$ the **replica trick** together with particular techniques which are available for integrable QFTs (e.g. form factor program)
- Main conclusion reached $\Rightarrow$ the leading correction to the entropy at large-distances is **largely model-independent**. More precisely, it depends only on the mass spectrum of the theory (for integrable QFT with diagonal S-matrix).

I. WHAT IS ENTANGLEMENT ENTROPY?

A measure of the **quantity of entanglement** between different parts of a quantum system.

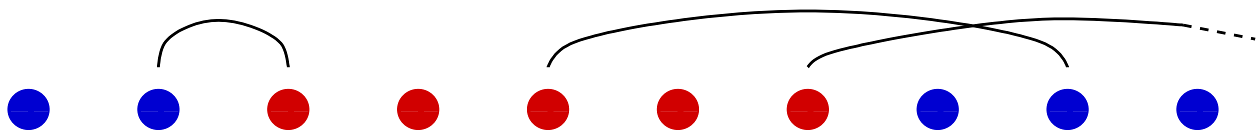
Entanglement: Characteristic of quantum systems whereby performing a local measurement may instantaneously effect local measurements far away.

- Reduced density matrix (in the ground state): $\rho_A = \mathrm{Tr}_{\bar{A}}(|\mathrm{gs}\rangle\langle\mathrm{gs}|)$



- (Bi-partite) Entanglement entropy: $S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A))$

It is the "number of links between A and $\bar{A}$ in the ground state" $\Rightarrow$ $S_A = S_{\bar{A}}$.



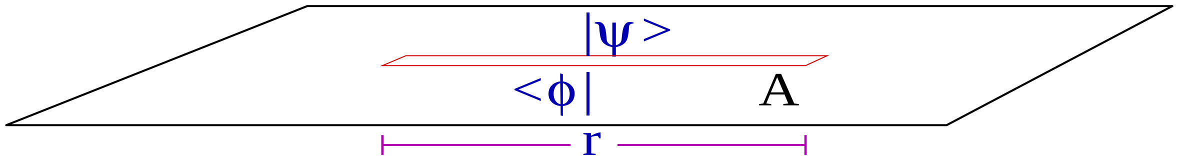
II. PARTITION FUNCTIONS ON MULTI-SHEETED RIEMANN SURFACES: SCALING LIMIT

- Scaling limit: correlation length $\xi \to \infty,\ L/\xi = mr$ fixed

QFT, mass m, lagrangian density $\mathcal{L}[\phi]$

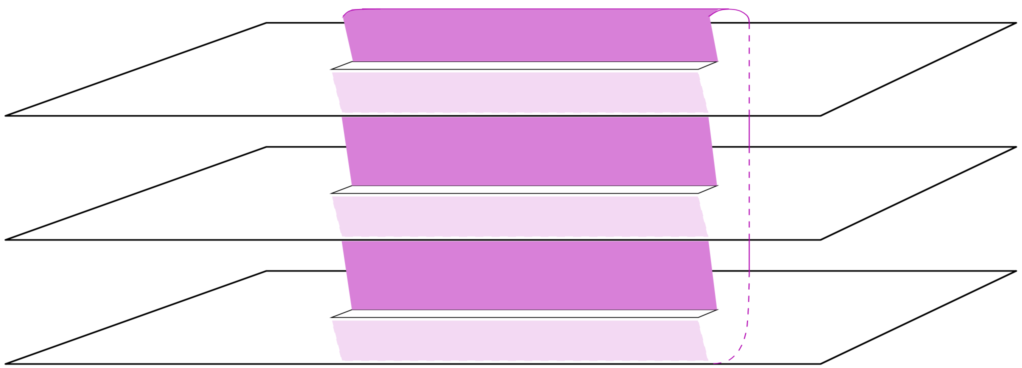
- "Replica trick:" $S_A = -\lim_{n\to 1} \frac{d}{dn} \mathrm{Tr}_A(\rho_A^n)$
- Partition function on Riemann surfaces for $n \in \mathbb{N}$ in the scaling limit:

$${}_A\langle \phi | \rho_A | \psi \rangle_A \quad \sim$$

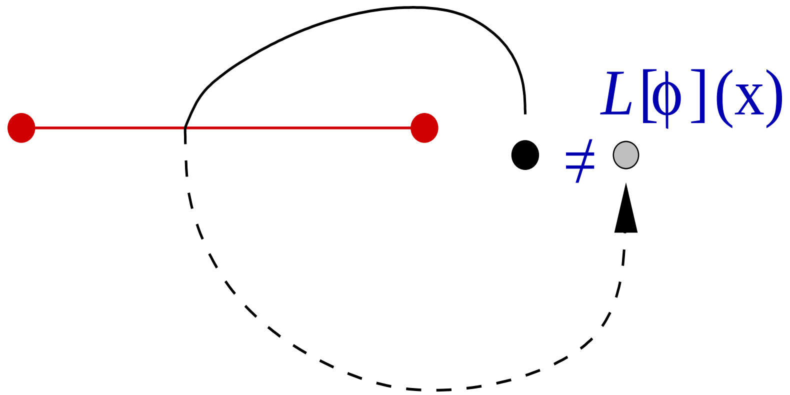


$$\mathrm{Tr}_A(\rho_A^n) \quad \sim \quad Z_n = \int [d\phi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\phi](x) \right]$$

$\mathcal{M}_3 \quad :$



Problem: Branch points correspond to non-local fields in the QFT $\mathcal{L}$



$$Z_n \not\propto \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}}$$

Solution: Branch-point twist fields

These are local twist fields associated to cyclic permutation symmetry of the n-copy model.

- Multi-copy model on $\mathbb{R}^2$:

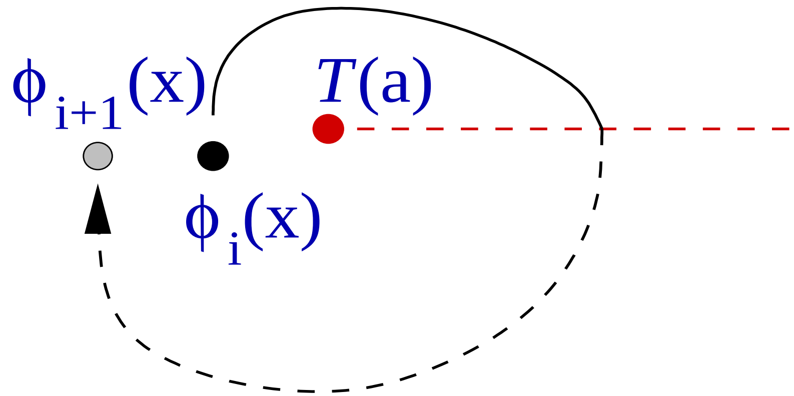
$$\mathcal{L}^{(n)}[\phi_1,\ldots,\phi_n](x) = \mathcal{L}[\phi_1](x) + \ldots + \mathcal{L}[\phi_n](x)$$

- Symmetry $\mathcal{L}^{(n)}[\sigma\phi_1,\ldots,\sigma\phi_n] = \mathcal{L}^{(n)}[\phi_1,\ldots,\phi_n]$, with $\sigma\phi_i = \phi_{i+1 \bmod n}$

- Associated twist fields $\mathcal{T}$:

$$\langle \mathcal{T}(a)\cdots\rangle_{\mathcal{L}^{(n)}} \propto \int_{C_a} [d\phi_1\cdots d\phi_n]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} \mathcal{L}^{(n)}[\phi_1,\ldots,\phi_n](x)\right]$$

C_a:

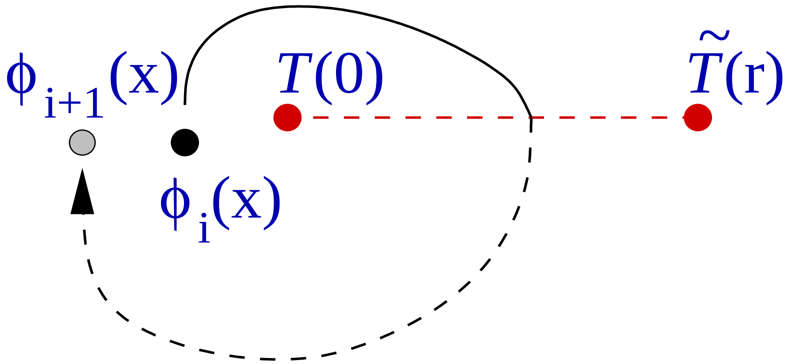


The main advantage is that branch points are now local fields in the QFT $\mathcal{L}^{(n)}$

Introducing a twist field $\tilde{\mathcal{T}}$ associated to the inverse symmetry σ^{-1} $\Rightarrow$ we obtain the partition function:

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \propto \int_{C_{0,r}} [d\phi_1 \cdots d\phi_n]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} \mathcal{L}^{(n)}[\phi_1,\ldots,\phi_n](x)\right] = Z_n$$

$C_{0,r}$:



$$\phi_i(y)\mathcal{T}(x) = \mathcal{T}(x)\phi_{i+1}(y) \qquad x^1 > y^1$$

$$\phi_i(y)\tilde{\mathcal{T}}(x) = \tilde{\mathcal{T}}(x)\phi_{i-1}(y) \qquad x^1 > y^1$$

III. SHORT- AND LARGE-DISTANCE ENTANGLEMENT ENTROPY

$$Z_n = \varepsilon^{2d_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \ , \quad S_A = -\lim_{n\to 1} \frac{d}{dn} Z_n$$

where ε is a non-universal short-distance cutoff and d_n is the scaling dimension of $\mathcal{T}$:

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

[Calabrese and Cardy, 2004]

- Short-distance: logarithmic behavior

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim r^{-2d_n} \ \Rightarrow \ S_A \sim -\frac{c}{3}\log\left(\frac{\varepsilon}{r}\right)$$

- Large-distance: saturation

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim \langle \mathcal{T}\rangle^2_{\mathcal{L}^{(n)}} \ \Rightarrow \ S_A \sim -\frac{c}{3}\log(m\varepsilon) - U$$

$$U = \frac{d}{dn}\left(m^{-2d_n}\langle \mathcal{T}\rangle^2_{\mathcal{L}^{(n)}}\right)\Big|_{n=1}$$

Our result: for any integrable QFT with diagonal scattering, the entropy with its **first correction to saturation** at large distances is:

$$S_A \sim -\frac{c}{3}\log(m\varepsilon) - U - \frac{1}{8}\sum_{\alpha=1}^{\ell} K_0(2rm_\alpha) + O\left(e^{-3rm_1}\right)$$

where ℓ is the number of particles in the spectrum of the QFT, and m_α are the masses of the particles, with $m_1 \leq m_\alpha \; \forall\alpha$.

IV. FORM FACTORS OF BRANCH-POINT TWIST FIELDS

For an integrable QFT $\mathcal{L}$ with a spectrum of one particle, no bound state, and S-matrix $S(\theta)$

- Scattering matrix of $\mathcal{L}^{(n)}$:

$$
\begin{aligned}
S_{ii}(\theta) &= S(\theta) \qquad \forall \quad i = 1, \ldots, n, \\
S_{ij}(\theta) &= 1, \qquad \forall \quad i, j = 1, \ldots, n \quad \text{and} \quad i \neq j,
\end{aligned}
$$

- Form factors of branch-point twist field in $\mathcal{L}^{(n)}$:

$$
F_k^{i_1 \ldots i_k}(\theta_1, \ldots, \theta_k) := \langle \mathrm{gs} | \mathcal{T}(0) | \theta_1, \ldots, \theta_k \rangle^{\text{in}}_{i_1, \ldots, i_k}
$$

$$
F_k^{\cdots \mu_i \mu_{i+1} \cdots}(\ldots, \theta_i, \theta_{i+1}, \ldots) = S_{\mu_i \mu_{i+1}}(\theta_i - \theta_{i+1}) F_k^{\cdots \mu_{i+1} \mu_i \cdots}(\ldots, \theta_{i+1}, \theta_i, \ldots)
$$

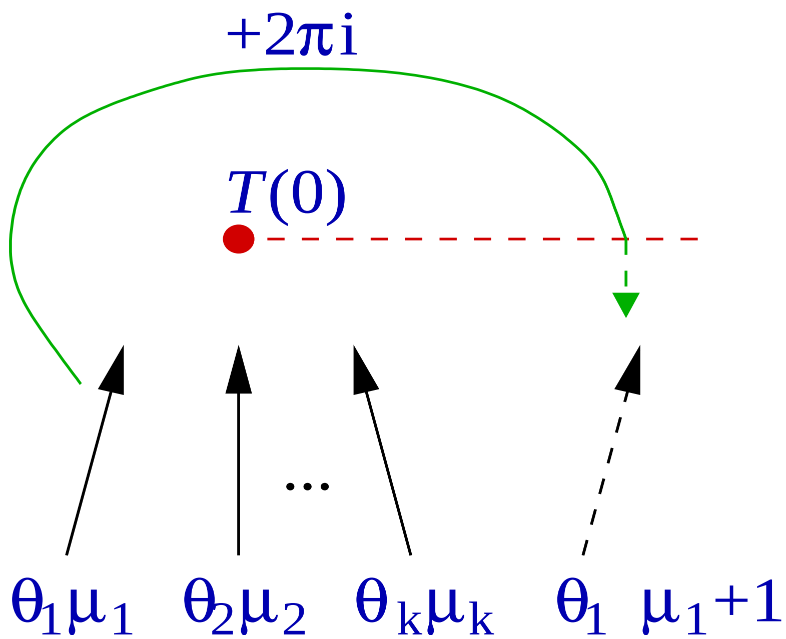
$$
F_k^{\mu_1 \mu_2 \ldots \mu_k}(\theta_1 + 2\pi i, \ldots, \theta_k) = F_k^{\mu_2 \ldots \mu_k \; \mu_1 + 1}(\theta_2, \ldots, \theta_k, \theta_1)
$$

$$
-i \text{Res}_{\bar{\theta}_0 = \theta_0} F_{k+2}^{\mu \mu \mu_1 \ldots \mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1 \ldots, \theta_k) = F_k^{\mu_1 \ldots \mu_k}(\theta_1, \ldots, \theta_k)
$$

$$
-i \text{Res}_{\bar{\theta}_0 = \theta_0} F_{k+2}^{\mu \; \mu+1 \; \mu_1 \ldots \mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1 \ldots, \theta_k) = -\prod_{i=1}^{k} S_{\mu \mu_i}(\theta_{0i}) F_k^{\mu_1 \ldots \mu_k}(\theta_1, \ldots, \theta_k)
$$

The quasi-periodicity relation

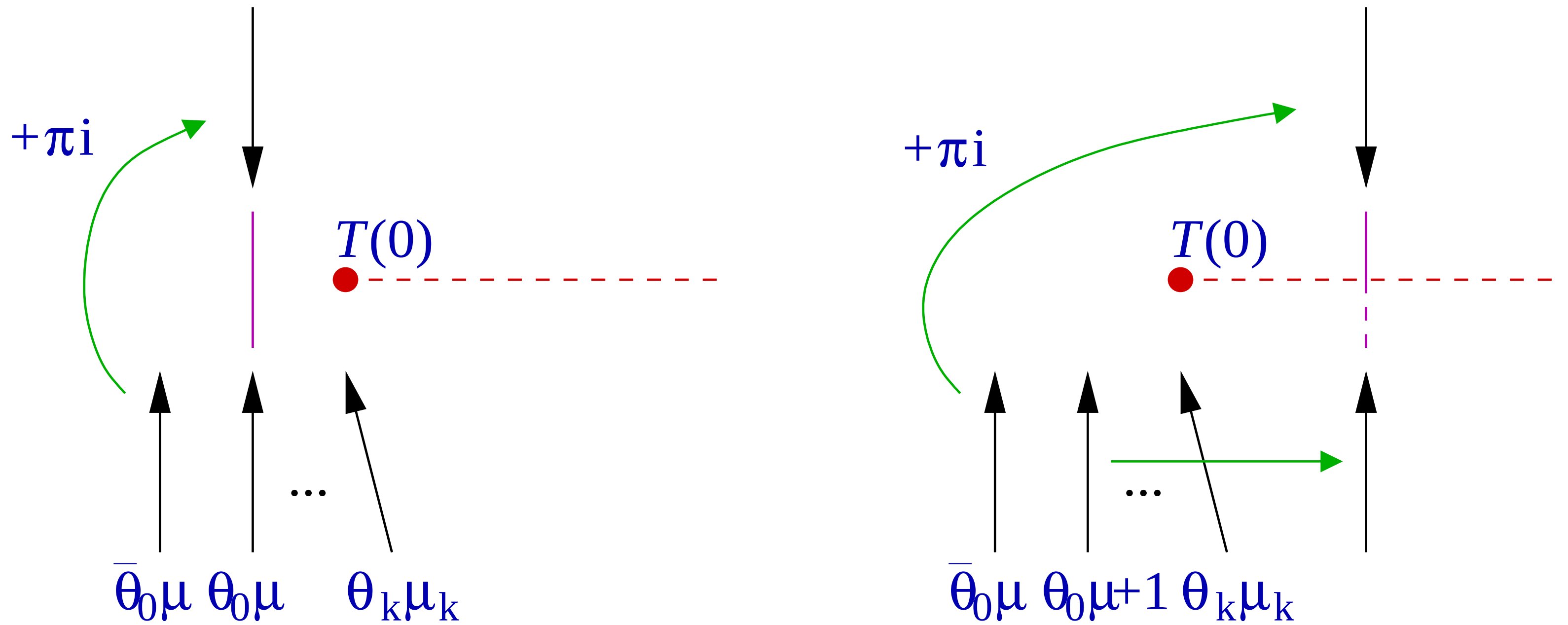
$$F_k^{\mu_1\mu_2\ldots\mu_k}(\theta_1 + 2\pi i, \ldots, \theta_k) = F_k^{\mu_2\ldots\mu_k\mu_1+1}(\theta_2, \ldots, \theta_k, \theta_1)$$



The kinematic residue equations

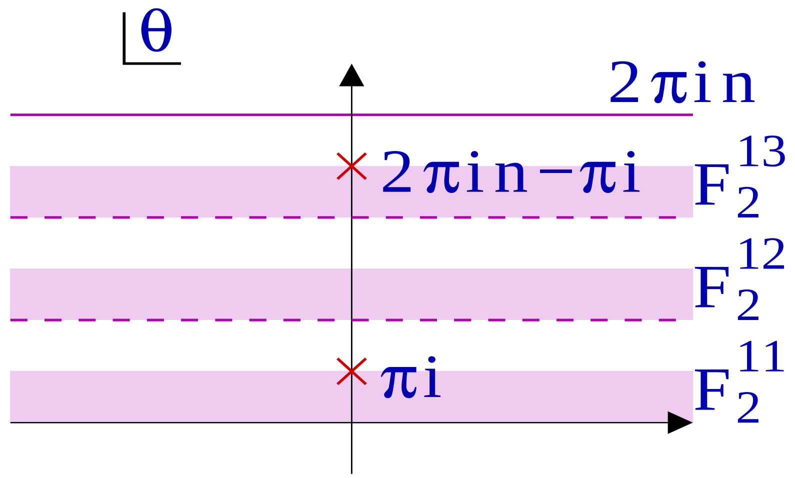
$$-i\mathrm{Res}_{\bar\theta_0=\theta_0} F_{k+2}^{\mu\mu\mu_1\ldots\mu_k}(\bar\theta_0+i\pi,\theta_0,\theta_1\ldots,\theta_k) = F_k^{\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)$$

$$i\mathrm{Res}_{\bar\theta_0=\theta_0} F_{k+2}^{\mu\ \mu+1\ \mu_1\ldots\mu_k}(\bar\theta_0+i\pi,\theta_0,\theta_1\ldots,\theta_k) = \prod_{i=1}^{k} S_{\mu\mu_i}(\theta_{0i}) F_k^{\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k)$$



The structure of the two-particle form factors

- **Basic properties:** $F_2^{ij}(\theta_1, \theta_2) = F_2^{1\;1+j-i}(\theta_1 - \theta_2)$
- **Only $F_2^{11}(\theta)$ matters:** $F_2^{1j}(\theta) = F_2^{11}(2\pi i(j-1) - \theta)\,, \quad j = 2, \ldots, n$
- **Non-trivial constraints:** $F_2^{11}(\theta) = S(\theta) F_2^{11}(-\theta) = F_2^{11}(2\pi i n - \theta)$



The exact two-particle form factors

With the integral representation for the scattering matrix:

$$S(\theta) = \exp\left[\int_0^\infty \frac{dt}{t} g(t) \sinh\left(\frac{t\theta}{i\pi}\right)\right]$$

the solution is

$$F_2^{11}(\theta) = \frac{\langle \mathcal{T} \rangle \sin\left(\frac{\pi}{n}\right)}{2n \sinh\left(\frac{i\pi-\theta}{2n}\right) \sinh\left(\frac{i\pi+\theta}{2n}\right)} \frac{F_{\mathsf{min}}^{11}(\theta)}{F_{\mathsf{min}}^{11}(i\pi)}$$

where

$$F_{\mathsf{min}}^{11}(\theta) = \exp\left[\int_0^\infty \frac{dt}{t \sinh(nt)} g(t) \sin\left(\frac{it}{2}\left(n + \frac{i\theta}{\pi}\right)\right)^2\right]$$

Ising and sinh-Gordon cases

- Ising case:

$$S(\theta) = -1\,, \quad F^{11}_{\text{min}}(\theta) = -i\sinh\frac{\theta}{2n}$$

- sinh-Gordon case:

$$S(\theta) = \frac{\tanh\frac{1}{2}\left(1-\frac{i\pi B}{2}\right)}{\tanh\frac{1}{2}\left(1+\frac{i\pi B}{2}\right)}\,, \quad g(t) = \frac{8\sinh\frac{tB}{4}\sinh\frac{t}{2}\left(1-\frac{B}{2}\right)\sinh\frac{t}{2}}{\sinh t}$$

Checks performed:

- Evaluating the scaling dimension using Delfino-Simonetti-Cardy formula [DSC'96] and Fring-Mussardo-Simonetti [FMS'93] form factors of the stress-energy tensor in sinh-Gordon: exact formula in the Ising case, good numerical accuracy for sinh-Gordon
- Evaluating the form factors directly in the angular quantisation using Brazhnikov-Lukyanov's [BL'98] angular quantisation for integrable models

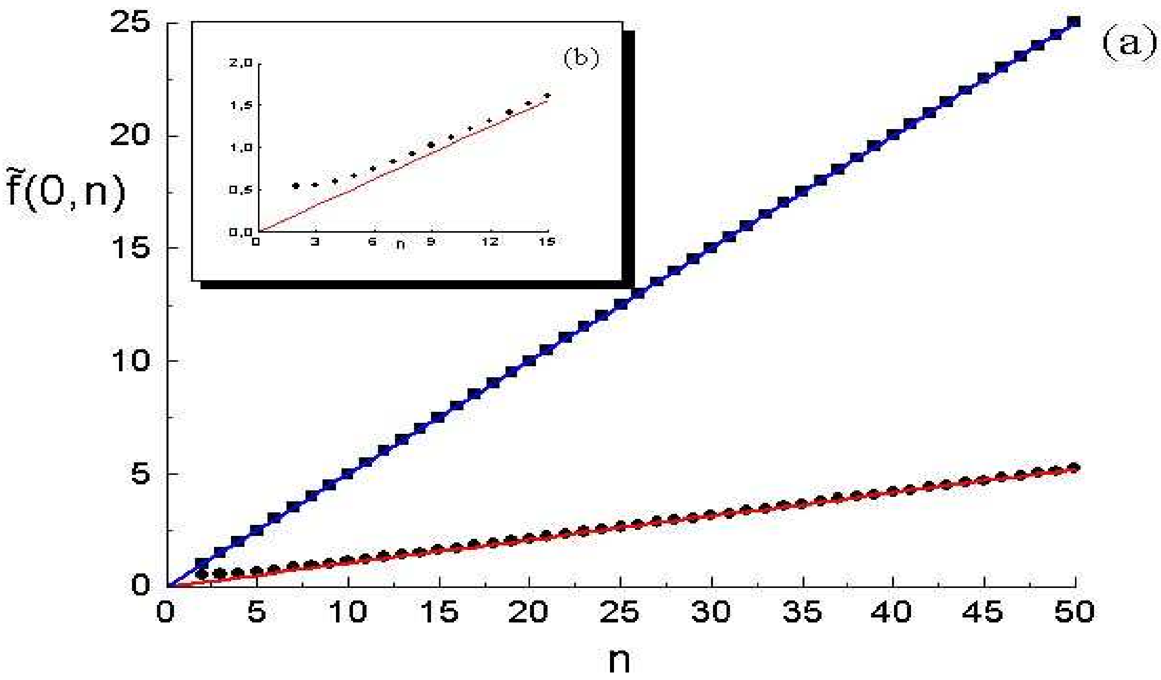
V. TWO-POINT CORRELATION FUNCTION OF TWIST FIELDS

$$\begin{aligned}
\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle &= \langle \mathrm{gs}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\
&= \sum_{\text{state } k} \langle \mathrm{gs}|\mathcal{T}(0)|k\rangle\langle k|\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\
&= \langle \mathcal{T}\rangle^2 + n\sum_{j=1}^{n}\int d\theta_1 d\theta_2 e^{-mr(\cosh\theta_1+\cosh\theta_2)}|F_2^{1j}(\theta_1-\theta_2)|^2 + \ldots \\
&= \langle \mathcal{T}\rangle^2 \left(1 + \frac{n}{4\pi^2}\int_{-\infty}^{\infty} f(\theta,n)K_0(2mr\cosh(\theta/2)d\theta + \ldots\right)
\end{aligned}$$

where

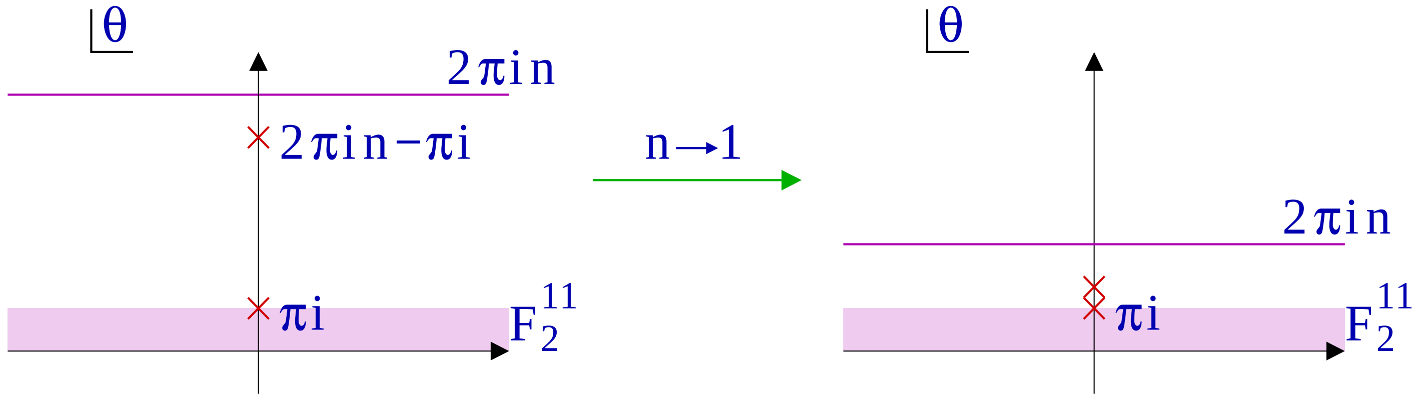
$$f(\theta,n) = \langle \mathcal{T}\rangle^{-2}\sum_{j=0}^{n-1}|F_2^{11}(-\theta+2\pi i j)|^2$$

In order to compute the entropy we would like to evaluate $\lim_{n\to 1} \frac{d}{dn}(nf(\theta,n)) \Rightarrow$ analytic continuation $\tilde{f}(\theta,n)$ of $f(\theta,n)$ from $n = 1, 2, 3, \ldots$ to $n \in [1,\infty)$



The analytic continuation $\tilde{f}(\theta,n)$ of $f(\theta,n)$ does not converge uniformly as $n \to 1$ on $\theta \in \mathbb{R}$, that is, $\tilde{f}(0,1) \neq f(0,1) = 0$

The non-zero value of $\tilde{f}(0,1)$ is due to the collision of poles of $|F_2^{11}(2\pi ij)|^2$ as function of j as $n \longrightarrow 1$, as can be seen from Poisson's re-summation formula



Poisson re-summation formula:

$$f(\theta,n) - s(\theta,0) = \sum_{j=1}^{n-1} s(\theta,j) = \sum_{k\in\mathbb{Z}} (s_{nk} - s_k)$$

$$s(\theta,j) = |F_2^{11}(-\theta + 2\pi ij)|^2 \,, \quad s_k = \int_0^n dj\; e^{-\frac{2\pi ijk}{n}} s(\theta,j)$$

Extracting the poles:

$$s(\theta, j) \sim \frac{iF_2^{11}(-2\theta + 2\pi i n - i\pi)}{-\theta - 2\pi i j + 2\pi i n - i\pi} - \frac{iF_2^{11}(-2\theta + i\pi)}{-\theta - 2\pi i j + i\pi} + \text{c.c.}$$

and re-summing them exactly gives

$$\tilde{f}(\theta, n) \sim \tilde{f}(0,1) \left(\frac{i\pi(n-1)}{2(\theta + i\pi(n-1))} - \frac{i\pi(n-1)}{2(\theta - i\pi(n-1))} \right) , \quad \tilde{f}(0,1) = \frac{1}{2}$$

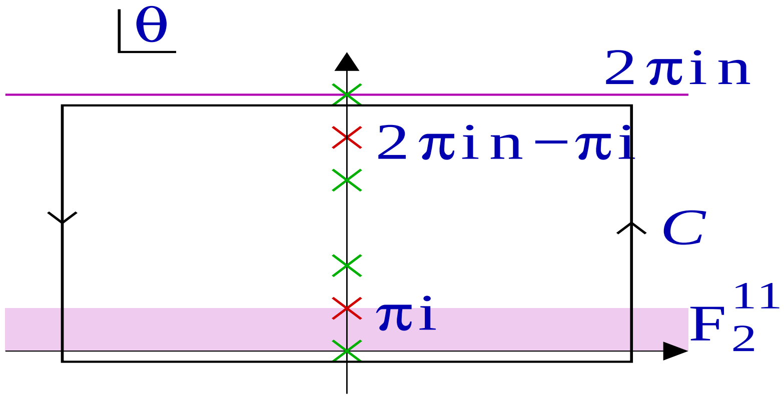
Hence the derivative is supported at $\theta = 0$:

$$\left(\frac{\partial}{\partial n} \tilde{f}(\theta, n) \right)_{n=1} = \pi^2 \tilde{f}(0,1) \delta(\theta)$$

There is an exact analytic continuation:

Consider the closed-contour integral

$$\int_{\mathcal{C}} \frac{dj}{2\pi i} \pi \cot \pi j \; F_2^{11}(2\pi i j)^2$$



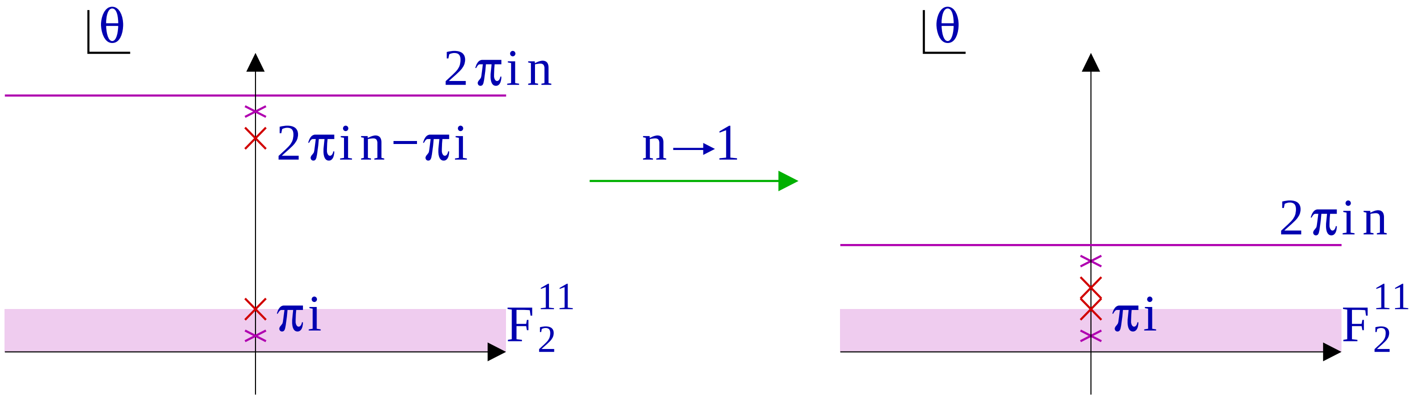
Assuming $F_2^{11}(0) = 0$ and $F_2^{11}(\theta) = 0$ at $|\theta| \to \infty$:

$$\tilde{f}(0,n) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Im}(S(-\theta)) \coth\left(\frac{\theta}{2}\right) |F_2^{11}(\theta)|^2 d\theta$$

VI. MULTI-PARTICLE AND BOUND-STATE CASE (DIAGONAL SCATTERING)

$$|\ldots,\theta_{\mu_i},\theta_{\mu_{i+1}},\ldots\rangle = S_{\mu_i\mu_{i+1}}|\ldots,\theta_{\mu_{i+1}},\theta_{\mu_i},\ldots\rangle\,, \quad \mu = (\text{type}, \text{sheet})$$

- For every particle type, there is a kinematic residue $\Rightarrow$ contribution at $n = 1$
- Possible bound states give additional poles on the physical sheet, on the imaginary line of θ, but they never collide $\Rightarrow$ no contribution at $n = 1$.

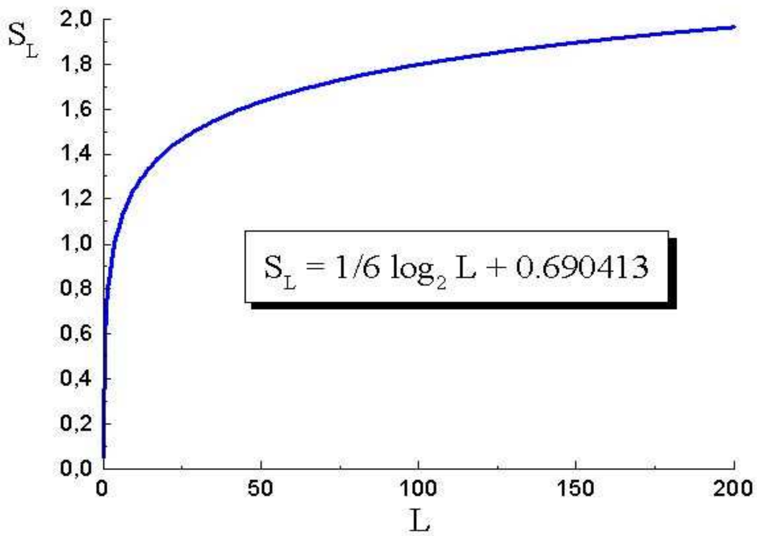


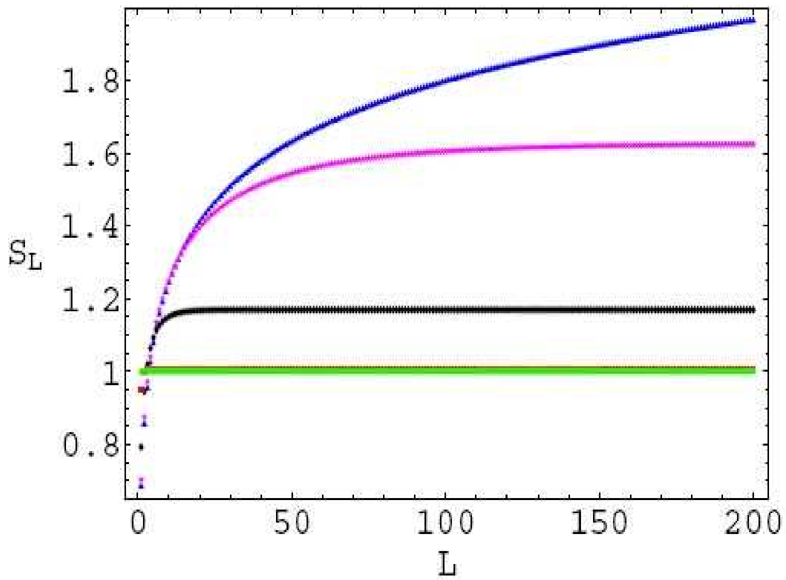
VII. EVALUATING THE SATURATION CONSTANT IN THE ISING MODEL

We evaluated the constant U for the Ising model, exploiting previous results for Ising one-point functions by Lukyanov and Zamolodchikov [LZ'97] and by relating the expectation value of the twist fields to that of free Fermions in the n-copy model.

We obtained $U_{\text{Ising}} = -0.131984...$

We have compared our results to numerical results in the Ising lattice by Latorre, Rico and Vidal [LRV'04]. We found very good agreement.





VIII. CONCLUSIONS

- The main result of this work is the derivation of the first correction to saturation of the entanglement entropy in any IQFT with diagonal scattering. The result is remarkably general. As a by-product of our investigation we obtained the saturation value of the entropy in the Ising model and derived the form factor equations associated to twist fields.
- There are several interesting open problems:
  - Generalisations to non-diagonal scattering should not be too difficult.
  - The evaluation of the higher-particle corrections to the entanglement entropy (implying the further development of the form factor program for twist fields) should be possible.
  - It would also be nice to compute the saturation value U for the sinh-Gordon model or other interacting theories.