International Workshop on Integrable Models and Applications

Boundary form factors of the sinh-Gordon model with Dirichlet boundary conditions at the self-dual point

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I. FORM FACTORS OF BULK THEORIES

- Form factors of IQFTs are defined as matrix elements of some local operator $\mathcal{O}$ located at the origin between a multi-particle *in*-state and the vacuum

$$F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n) := \langle 0|\mathcal{O}(0)Z_{\mu_1}(\theta_1)Z_{\mu_2}(\theta_2)\ldots Z_{\mu_n}(\theta_n)|0\rangle_{\text{in}}$$

  - ▷ $|0\rangle$ represents the vacuum state.
  - ▷ $Z_{\mu_i}(\theta_i)$ are vertex operators which generate the space of physical states.
  - ▷ μ_i are quantum numbers characterizing the various particle species.
  - ▷ The parameters θ_i are the rapidities.

- Over the last 30 years, the form factor program has been extensively and successfully employed for the computation of form factors, correlation functions and other quantities computable from the latter (e.g. UV central charge ...)

M. Karowski and P. Weisz, Nucl. Phys. **B139**, 455 (1978).

F. Smirnov, Form factors in completely integrable models of quantum field theory, Adv. Series in Math. Phys. **14**, World Scientific, Singapore (1992).

II. FORM FACTORS OF BOUNDARY THEORIES I

- Can the form factor program be extended to integrable theories with boundaries or defects? Yes, this may be done in two possible ways:

▷ boundary along the space direction at zero value of the time coordinate: the Hilbert space is the same as for the bulk theory and the presence of the boundary is characterized by a boundary state

S. Ghoshal and A. B. Zamolodchikov, Int. J. Mod. Phys. **A9**, 3841 (1994).

$$|B\rangle := \exp\left(\frac{1}{4\pi}\int_{-\infty}^{\infty} R_{\mu_i}(\frac{i\pi}{2}-\theta)Z_{\mu_i}(-\theta)Z_{\mu_i}(\theta)d\theta\right)|0\rangle.$$

- Here $R_{\mu_i}(\theta)$ is the reflection amplitude off the boundary of particle μ_i.
- In this case boundary form factors can be expressed in terms of bulk form factors (usually they will be an infinite sum of the latter).

G. Delfino, G. Mussardo and P. Simonetti, Nucl. Phys. **B432**, 518 (1994) [defects]

R. Konik, A. LeClair and G. Mussardo, Int. J. Mod. Phys. **A11**, 2765 (1996) [boundaries]

O. A. Castro-Alvaredo and A. Fring, Nucl. Phys. **B649 [FS]**, 449 (2003) [defects]

Z. Bajnok and A. George, Int. J. Mod. Phys. **A21**, 1063 (2006) [defects and boundaries]

III. FORM FACTORS OF BOUNDARY THEORIES II

▷ boundary along the time direction at zero value of the space coordinate:
the Hilbert space will change with respect to the bulk case, as only half of the physical states remain linearly independent thanks to

$$Z_{\mu_i}(\theta_i) = R_{\mu_i}(\theta_i) Z_{\mu_i}(-\theta_i)$$

- In this case the recent work

Z. Bajnok, L. Palla, G. Takacs, Nucl. Phys. **B750** 179 (2006).

has established that a set of consistency equations need to be satisfied by the form factors of boundary operators. These equations may be solved in a similar way as for the bulk, so that we can speak of a boundary form factor program.

- The work I intend to present today builds on the results already obtained in the previous paper for a particular boundary theory, extending them and culminating with a conjecture for the structure of all n-particle form factors of boundary operators in the model.

IV. BOUNDARY FORM FACTOR AXIOMS

- For a theory with diagonal scattering, a single particle type and where neither the S-matrix nor the reflection amplitudes have poles in the physical sheet the axioms are [Z. Bajnok, L. Palla, G. Takacs, Nucl. Phys. **B750** 179 (2006)]:

  ▷ From the braiding relations of operators $Z_i(\theta_i)$ and B:

$$
\begin{aligned}
F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_i,\theta_{i+1},\ldots\theta_n) &= S(\theta_{i\,i+1})F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots\theta_n)\\
F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_{n-1},\theta_n) &= R(\theta_n)F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_{n-1},-\theta_n)
\end{aligned}
$$

  ▷ From crossing symmetry:

$$
F_n^{\mathcal{O}}(\theta_1,\theta_2,\ldots,\theta_n) = R(i\pi-\theta_1)F_n^{\mathcal{O}}(2\pi i-\theta_1,\theta_2,\ldots,\theta_n)
$$

  ▷ Kinematical residue equation

$$
-i \underset{\bar{\theta}_0=\theta_0}{\mathrm{Res}}\; F_{n+2}^{\mathcal{O}}(\theta_0+i\pi,\bar{\theta}_0,\theta_1\ldots,\theta_n) = \left(1-\prod_{i=1}^{n} S(\theta_{0i})S(\hat{\theta}_{0i})\right)F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)
$$

- with $\theta_{ij} := \theta_i-\theta_j$ and $\hat{\theta}_{ij} := \theta_i+\theta_j$.

V. THE SOLUTION PROCEDURE

- The solution procedure of these equations resembles that developed for bulk theories: first, an ansatz can be made which automatically solves the first 3 equations [Z. Bajnok, L. Palla, G. Takacs, Nucl. Phys. **B750** 179 (2006)]:

$$F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n) = H_n^{\mathcal{O}} Q_n^{\mathcal{O}}(y_1,\ldots,y_n) \prod_{i=1}^{n} r(\theta_i) \prod_{1\leq i<j\leq n} \frac{f(\theta_{ij}) f(\hat{\theta}_{ij})}{y_i + y_j}.$$

  - ▷ $y_i = e^{\theta_i} + e^{-\theta_i}$ and $H_n^{\mathcal{O}}$ is a constant.
  - ▷ $Q_n^{\mathcal{O}}(y_1,\ldots,y_n)$ is an entire function of the variables $y_1,\ldots,y_n$.
  - ▷ $f(\theta)$ is the same minimal (2-particle) form factor found in the bulk case.
  - ▷ $r(\theta)$ is the minimal 1-particle form factor.

- Integral representations for these minimal form factors can be obtained from the representations of the S-matrix and reflection amplitude $R(\theta)$.
- When this ansatz is plugged into the kinematic residue equation a recursive relation between $Q_{n+2}^{\mathcal{O}}$ and $Q_n^{\mathcal{O}}$ is obtained.

VI. THE MODEL

- In this work we have considered the sinh-Gordon model with particular Dirichlet boundary conditions.
- The bulk S-matrix and reflection amplitude of the model are given by

I.Ya. Arafeva and V.E. Korepin, Pis'ma Zh. Eksp. Teor. Fiz. **20**, 680 (1974) [S-matrix].

S. Ghoshal, Int. J. Mod. Phys. **A9**, 4801 (1994) [R-matrix].

E. Corrigan and A. Taormina, J. Phys. **A33**, 8739 (2000) [R-matrix].

$$S(\theta) = -(-B)_\theta (B-2)_\theta, \quad R(\theta) = (1)_\theta \left(1 + \frac{B}{2}\right)_\theta \left(2 - \frac{B}{2}\right)_\theta \frac{(E-1)_\theta}{(E+1)_\theta}$$

in terms of the building blocks

$$(x)_\theta = \frac{\sinh \frac{1}{2}\left(\theta + \frac{i\pi x}{2}\right)}{\sinh \frac{1}{2}\left(\theta - \frac{i\pi x}{2}\right)}$$

- In our work we have studied a particularly simple situation: the self-dual point $B = 1$ and $E = 0$ which corresponds to vanishing value of the sinh-Gordon field at the boundary.

VII. THE FORM FACTOR RECURSIVE EQUATIONS

- The recursive equations for this particular model were already obtained by Bajnok et al. (2006) for generic values of B. They are

$$Q^{\mathcal{O}}_{n+2}(-y, y, y_1, \ldots, y_n) = -P_n(y, y_1, \ldots, y_n) Q^{\mathcal{O}}_n(y_1, \ldots, y_n).$$

with

$$P_n(y, y_1, \ldots, y_n) = \frac{i(-\omega_+ \omega_-)^n}{\omega_+ - \omega_-} \sum_{k,p=0}^{n} (-i\omega_-)^{-k} (-i\omega_+)^{-p} \, \sigma^n_k \sigma^n_p \sin(\frac{\pi(k-p)}{2})$$

in terms of elementary symmetric polynomials σ^n_k

$$\prod_{k=1}^{n} (z + y_k) = \sum_{k=0}^{n} z^{n-k} \sigma^n_k,$$

and the variables

$$\omega_{\pm} = 2\cosh(\theta \pm i\pi B/2), \qquad y = e^{\theta} + e^{-\theta}$$

VIII. NEW FEATURES AT $B = 1$

- At $B = 1$ a factorized structure both of the recursive equations and their solutions appears which is absent for generic values of B: the parameters $\omega_\pm$

$$\omega_\pm = \pm 2i \sinh\theta$$

become proportional to each other and this allows us to bring P_n into the factorized form

$$P_n(y, y_1, \ldots, y_n) = P_n^e(y, y_1, \ldots, y_n) P_n^o(y, y_1, \ldots, y_n)$$

with

$$\begin{aligned} P_n^e(y, y_1, \ldots, y_n) &= (-i\omega_+)^{2\left[\frac{n}{2}\right]} \sum_{p=0}^{n} \sigma_p^n (-i\omega_+)^{-p} \cos(\frac{\pi p}{2}) \\ P_n^o(y, y_1, \ldots, y_n) &= (-1)^{n+1} (-i\omega_+)^{2\left[\frac{n+1}{2}\right]-1} \sum_{p=0}^{n} \sigma_p^n (-i\omega_+)^{-p} \sin(\frac{\pi p}{2}) \end{aligned}$$

- Here $[x]$ indicates the integer part of x.

IX. FACTORIZED SOLUTIONS

- As mentioned before, at $B = 1$ we have observed that also the solutions of the recursive equations, $Q_n^{\mathcal{O}}$ factorize.
- As a consequence of this factorization the recursive equation above can be split in two independent equations:

$$\begin{aligned} Q_{n+2}^{\mathcal{O}|e}(-y, y, y_1, \ldots, y_n) &= P_n^e(y, y_1, \ldots, y_n) Q_n^{\mathcal{O}|e}(y_1, \ldots, y_n) \\ Q_{n+2}^{\mathcal{O}|o}(-y, y, y_1, \ldots, y_n) &= -P_n^o(y, y_1, \ldots, y_n) Q_n^{\mathcal{O}|o}(y_1, \ldots, y_n) \end{aligned}$$

with

$$Q_n^{\mathcal{O}}(y_1, \ldots, y_n) = Q_n^{\mathcal{O}|e}(y_1, \ldots, y_n) Q_n^{\mathcal{O}|o}(y_1, \ldots, y_n)$$

- Solving the two recursive equations above is in general much simpler than solving the initial equation directly. This has allowed us to construct solutions up to very high particle numbers ($n = 16$) and to make a conjecture for the general structure of these solutions.

X. THE FORM FACTORS OF $\partial_x\phi$ AND $(\partial_x\phi)^2$

- In the work of Bajnok et al. (2006) the first form factors (up to $n = 5$) of two different operators were constructed. On the basis of their asymptotic behavior, the operators were identified as $\partial_x\phi$ and $(\partial_x\phi)^2$.
- For symmetry reasons $\partial_x\phi$ and $(\partial_x\phi)^2$ have only non-vanishing form factors for odd and even particle numbers, respectively.
- Notice that with Dirichlet boundary conditions the boundary value of ϕ is constant (in this case zero) and therefore the operator content consists solely of space derivatives of ϕ.

$$
\begin{aligned}
Q_1^{\partial_x\phi|e} &= 1, & Q_1^{\partial_x\phi|o} &= 1 \\
Q_2^{(\partial_x\phi)^2|e} &= -1, & Q_2^{(\partial_x\phi)^2|o} &= \sigma_1 \\
Q_3^{\partial_x\phi|e} &= 1, & Q_3^{\partial_x\phi|o} &= -\sigma_1 \\
Q_4^{(\partial_x\phi)^2|e} &= 4+\sigma_2, & Q_4^{(\partial_x\phi)^2|o} &= \sigma_1^2 \\
Q_5^{\partial_x\phi|e} &= -(4+\sigma_2), & Q_5^{\partial_x\phi|o} &= -\sigma_1(4\sigma_1+\sigma_3)
\end{aligned}
$$

$$
Q_6^{(\partial_x\phi)^2|e} = (4+\sigma_2)(4(4+\sigma_2)+\sigma_4)-\sigma_6, \qquad Q_6^{(\partial_x\phi)^2|o} = -\sigma_1^2(4\sigma_1+\sigma_3)
$$

XI. SOME PATTERNS

- From the previous expressions various interesting patterns can be identified:

▷ All form factors of $\partial_x\phi$ can be obtained from those of $(\partial_x\phi)^2$:

$$
\begin{aligned}
Q_{2n}^{(\partial_x\phi)^2|e} &= -Q_{2n+1}^{\partial_x\phi|e}, \quad \text{for} \quad n\in\mathbb{Z}^+, \\
Q_{2n}^{(\partial_x\phi)^2|o} &= -\sigma_1 Q_{2n-1}^{\partial_x\phi|o}, \quad \text{for} \quad n\in\mathbb{Z}^+.
\end{aligned}
$$

These relations are only true in terms of elementary symmetric polynomials!

▷ The functions $Q_{2n+2}^{(\partial_x\phi)^2|o}$ can always be obtained from $Q_{2n}^{(\partial_x\phi)^2|e}$.

$$
Q_4^{(\partial_x\phi)^2|e} = 4+\sigma_2, \qquad Q_6^{(\partial_x\phi)^2|o} = -\sigma_1^2(4\sigma_1+\sigma_3)
$$

- The rule can be summarized in two steps:

▷ σ_{2k} in $Q_{2n}^{(\partial_x\phi)^2|e}$ should be replaced by σ_{2k+1} in $Q_{2n+2}^{(\partial_x\phi)^2|o}$.

▷ Every term in $Q_{2n+2}^{(\partial_x\phi)^2|o}$ should then be multiplied by as many factors σ_1 as to achieve that every term is a product of $n+1$ symmetric polynomials.

XII. A CONJECTURE

- According to the previous discussion the knowledge of the polynomials $Q_{2n}^{(\partial_x\phi)^2|e}$ is sufficient to characterize all form factors of $(\partial_x\phi)^2$ and $\partial_x\phi$.
- A conjecture for their general structure, based on numerous case-by-case studies, has been made (a proof is in progress):

$$Q_{2n}^{(\partial_x\phi)^2|e} = (-1)^{n+1} Q_{2n-2}^{(\partial_x\phi)^2|e} \mu_{2n-2} + \sigma_{2n} R_{2n}^e$$

▷ μ_i are linear combinations of elementary symmetric polynomials:

$$\mu_i = \sum_{k=0}^{[i/2]} 4^k \sigma_{i-2k}, \quad \text{e.g.} \quad \mu_1 = \sigma_1, \quad \mu_2 = 4+\sigma_2, \ldots, \mu_n = 4\mu_{n-2}+\sigma_n$$

▷ The part $(-1)^{n+1} Q_{2n-2}^{(\partial_x\phi)^2|e} \mu_{2n-2}$ does not contain the maximum degree polynomial σ_{2n}.

▷ The part $\sigma_{2n} R_{2n}^e$ includes all terms proportional to σ_{2n}.

▷ R_{2n}^e is a generally unknown combination of symmetric polynomials.

XIII. SOME EXAMPLES

$$\begin{aligned}
Q_2^{(\partial_x\phi)^2|e} &= -1, \\
Q_4^{(\partial_x\phi)^2|e} &= \underbrace{4+\sigma_2}_{\mu_2}, \\
Q_6^{(\partial_x\phi)^2|e} &= \underbrace{(4+\sigma_2)}_{Q_4^{(\partial_x\phi)^2|e}}\big(\underbrace{4(4+\sigma_2)+\sigma_4}_{\mu_4}\big)\underbrace{-\sigma_6}_{\sigma_6 R_6^e}, \\
Q_8^{(\partial_x\phi)^2|e} &= -\underbrace{(\mu_2\mu_4-\sigma_6)}_{Q_6^{(\partial_x\phi)^2|e}}\underbrace{(4\mu_4+\sigma_6)}_{\mu_6}-\underbrace{(8+\sigma_2)^2}_{R_8^e}\sigma_8.
\end{aligned}$$

XIV. SOLUTIONS FOR OTHER LOCAL FIELDS: DESCENDENT FIELDS I

- We have also found form factor solutions for fields other than $\partial_x \phi$ and $(\partial_x \phi)^2$.
- For this we proceed as in the works

J. L. Cardy and G. Mussardo, Nucl. Phys. **B340**, 387 (1990).[Ising]

P. Christe, Int. J. Mod. Phys. **A6**, 5271 (1991). [Yang-Lee]

A. Fring, G. Mussardo, and P. Simonetti, Nucl. Phys. **B393**, 413 (1993). [sinh-Gordon]

and we look for solutions of the form factor equations of the form

$$F_n^{\mathcal{O}'}(y_1, \ldots, y_n) = I_n^s(y_1, \ldots, y_n) F_n^{\mathcal{O}}(y_1, \ldots, y_n)$$

▷ $F_n^{\mathcal{O}}(y_1, \ldots, y_n)$ is a known solution of the form factor equations.

▷ $I_n^s(y_1, \ldots, y_n)$ is an entire function such that $F_n^{\mathcal{O}'}(y_1, \ldots, y_n)$ also satisfies the form factor equations.

▷ Here the index s characterizes the asymptotic behaviour of I_n^s (in bulk theories it is the spin of $\mathcal{O}'$).

XV. SOLUTIONS FOR OTHER LOCAL FIELDS: DESCENDENT FIELDS II

- $F_n^{\mathcal{O}'}(y_1, \ldots, y_n)$ satisfies all form factor consistency equations if:
  - ▷ I_n^s is a function of the variables $y_1, \ldots, y_n$.
  - ▷ I_n^s is an entire function which satisfies the recursive equation

$$I_{n+2}^s(-y, y, y_1, \ldots, y_n) = I_n^s(y_1, \ldots, y_n)$$

- This equation is exactly the same found in the study of bulk theories (up to replacement of variables y_i by $x_i = e^{\theta_i}$). It is solved by

$$I_n^{2s-1} = (-1)^{s+1} \det \mathcal{I}$$

with $\mathcal{I}$ being a matrix of entries

$$\mathcal{I}_{1j} = \sigma_{2j-1} \qquad \text{and} \qquad \mathcal{I}_{ij} = \sigma_{2j-2i+2},$$

for $j = 1, \ldots, s$ and $i = 2, \ldots, s$.

- It is natural to think that these new fields are in one-to-one correspondence with descendent fields at conformal level. However this still needs to be proven.

XVI. A SUMMARY OF THE MAIN RESULTS

- This work takes as starting point the form factor program for boundary integrable models proposed in Z. Bajnok, L. Palla and G. Takacs, Nucl. Phys. **B750** 179 (2006) and the results obtained there for the boundary sinh-Gordon model.

- The main result of this work has been a conjecture for the structure of all form factors of the sinh-Gordon model at the self dual point ($B = 1$) with particular ($\phi(0) = 0$) Dirichlet boundary conditions.

- The conjecture is based on the computation of form factor solutions up to very high particle numbers ($n = 16$) for the operators $\partial_x \phi$ and $(\partial_x \phi)^2$.

- The identification of various distinguished features of the form factor solutions (like their factorization into “even” and “odd” parts) which only seem to occur at $B = 1$ has been crucial for our computations.

- We also found form factor solutions for operators other than $\partial_x \phi$ and $(\partial_x \phi)^2$, which we expect to be related to descendent fields at conformal level.

XVII. OUTLOOK

● There are several problems which we would like to investigate in the future:

▷ To find a rigorous proof of the formulae conjectured for the form factors of $\partial_x \phi$ and $(\partial_x \phi)^2$.

▷ To investigate whether or not this kind of structure also appears for other models: some progress has been made in this direction and indeed similar formulae can be found for generic values of B and for the Yang-Lee model.

▷ To investigate the correspondence between our solutions and fields at conformal level along the lines of the original study performed for the Ising model (Cardy et al. (1990)).

▷ To carry out the boundary form factor program for other models.

▷ To employ the form factors obtained for the computation of correlation functions of boundary fields and other quantities related to them (some work has already been done by Bajnok et al. (2006)).