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# Form Factors from Algebraic Bethe Ansatz for Higher Spin Representations

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## I. FORM FACTORS AND CORRELATION FUNCTIONS OF QUANTUM SPIN CHAINS

● In the context of quantum spin chains, there are today two different approaches which allow to compute form factors and correlation functions of quantum spin chains (the examples treated are mostly the spin 1/2 XXX and XXZ spin chains):

$$\langle \mathcal{O} \rangle = \frac{\mathrm{tr}_{\mathcal{H}}(\mathcal{O} e^{-H/kT})}{\mathrm{tr}_{\mathcal{H}}(e^{-H/kT})} \quad \underset{T \to 0}{\Rightarrow} \quad \frac{\langle \Psi_g | \mathcal{O} | \Psi_g \rangle}{\langle \Psi_g | \Psi_g \rangle}$$

● In the first approach form factors and correlation functions are described in terms of $q$-deformed vertex operators and can be obtained as solutions to q-deformed KZ equations

M. Jimbo, T. Miwa et al. (1992-1996) [integral representations for the n-point correlations for the XXZ spin chain].

H.E. Boos, V.E. Korepin (2001); H.E. Boos, V.E. Korepin, Y. Nishiyama and M. Shiroishi (2002); H.E. Boos, V.E. Korepin and F.A. Smirnov (2003); K. Sakai, M. Shiroishi, Y. Nishiyama and M. Takahashi (2003); G. Kato, M. Shiroishi, M. Takahashi and K. Sakai (2003,2004); M. Takahashi, G. Kato and M. Shiroishi (2004); H. Boos, M. Jimbo, T. Miwa, F. Smirnov and Y. Takeyama (2004,2005) [formulae in terms of $\zeta$-functions for correlation functions of the XXX and XXZ chains (initially for 2th, 3th and 4th-neighbour correlations)].

$\Rightarrow$ I will not use this approach here!

• The other approach combines the algebraic Bethe ansatz technique (which provides a construction scheme for the quantum states) and the solution of the inverse scattering problem (which allows to write local operators on the chain in terms of the same objects the quantum states are made of).

• Once quantum states and operators are written in terms of the same operators $\{A, B, C, D\}$, correlation functions and form factors can be computed explicitly by exploiting the commutation relations amongst these objects.

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N. Kitanine, J.-M. Maillet and V. Terras (1999); J.-M. Maillet and V. Terras (2000) [solution of the inverse scattering problem]. N. Kitanine, J.-M. Maillet, N.A. Slavnov and V. Terras (1999-2005) [integral representations for correlation functions and form factors of the spin 1/2 XXZ chain (dynamical correlation functions, roots of unity...)]. N. Kitanine (2001) [correlation functions for the spin $s$ XXX chain]. J.-S-Caux and J.-M. Maillet (2005); J.-S-Caux, R. Hagemans and J.-M. Maillet (2005) [numerical applications]. F. Göhmann, A. Klümper and A. Seel (2004); M. Bortz, F. Göhmann (2005); F. Göhmann and A. Seel (2005) [integral representations for correlation functions at finite T]

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• Here we will be using this technique for the first time for mixed spin chains (different spin representations at different sites), e.g. impurity systems and alternating spin chains.

• The main result of this work are closed determinant expressions for the form factors of $\{S^z, S^\pm\}$ in arbitrary spin representations and for arbitrary spins at other sites of the chain.

## II. ALGEBRAIC BETHE ANSATZ

- The algebraic Bethe ansatz technique provides a closed algebraic setup which allows the simultaneous construction of the conserved charges, e.g. **Hamiltonian**, and of its **eigenstates**.

L.D Faddeev, E.K. Sklyanin and L.A. Takhtajan (1979).

$$\underbrace{R^{(\frac{1}{2},\frac{1}{2})}(\lambda)}_{\mathbb{C}^2\otimes\mathbb{C}^2} = \begin{pmatrix} \frac{\varphi\left(\lambda+\frac{\eta}{2}(\sigma^z+1)\right)}{\varphi(\lambda+\eta)} & \frac{\varphi(\eta)}{\varphi(\lambda+\eta)}\sigma^- \\ \frac{\varphi(\eta)}{\varphi(\lambda+\eta)}\sigma^+ & \frac{\varphi\left(\lambda+\frac{\eta}{2}(1-\sigma^z)\right)}{\varphi(\lambda+\eta)} \end{pmatrix}$$

$$\varphi(\lambda) = \begin{cases} \lambda & \text{for the XXX chain} \\ \sinh(\lambda) & \text{for the XXZ chain; } [\Delta = \cosh\eta] \end{cases}$$

- The $R$-matrix is a solution of the Yang-Baxter equations (integrability). The **quantum monodromy matrix** is then

$$T^{(\frac{1}{2})}_{0;1\ldots N}(\lambda;\{\xi\}) = \underbrace{R^{(\frac{1}{2},s_N)}_{0N}(\lambda-\xi_N)\cdots R^{(\frac{1}{2},s_1)}_{01}(\lambda-\xi_1)}_{\text{we want to look at mixed chains!}}$$

$$= \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}; \quad t^{(1/2)}(\lambda,\{\xi\}) = (A+D)(\lambda)$$

- $\xi_1\ldots\xi_N$ are the **inhomogeneity parameters**. Their presence simplifies certain computations. Ultimately $\xi_i = \eta/2\,\forall\, i$.

- The transfer matrix $t^{(1/2)}(\lambda,\{\xi\})$ generates the Hamiltonian of the model

$$\boxed{H \sim \left.\frac{d\log(t^{(1/2)}(\lambda,\{\xi\}))}{d\lambda}\right|_{\lambda=\eta/2=\xi_1\ldots\xi_N}}$$

- Since $[H,t(\lambda)]=0$, the eigenstates of $H$ are those of $t(\lambda)$ and have the form

$$|\Psi(\{\lambda\})\rangle = B(\lambda_1)\cdots B(\lambda_\ell)|0\rangle \quad \text{with}$$

$$\prod_{k=1}^{\ell}\frac{\varphi(\lambda_k-\lambda_j+\eta)}{\varphi(\lambda_k-\lambda_j-\eta)} = -d(\lambda_j) \quad \text{BA-equations}$$

- Here $|0\rangle$ denotes the completely ferromagnetic reference state (all spins up) and $d(\lambda)$ is the eigenvalue of $D(\lambda)$ on that state

$$D(\lambda)|0\rangle = d(\lambda)|0\rangle = \underbrace{\prod_{j=1}^{N}\frac{\varphi(\lambda-\xi_j-(s_j-1/2)\eta)}{\varphi(\lambda-\xi_j+(s_j+1/2)\eta)}}_{\text{Eigenvalue of } \exp(-ip_{\text{tot}}(\lambda))}|0\rangle$$

- The knowledge of the ground state is the first step for the computation of correlation functions and form factors.

# III.THE INVERSE SCATTERING PROBLEM

- The next step in our problem is that of finding a realization of states and fields in terms of the same objects: $\{A, B, C, D\}$.

J.M. Maillet, V. Terras and N. Kitanine (1999);J.M. Maillet and V. Terras (2000).

- The previous works were the starting point of a program leading to the exact computation of many correlation functions and form factors of the XXX and XXZ chains, mostly for spin 1/2.
- The solution of the inverse problem for the generators of the $su(2)$ algebra $S_j^z, S_j^\pm$ is

$$S_j^{z,\pm} = \left[\prod_{k=1}^{j-1} t^{(s_k)}(\xi_k)\right] \Lambda_{z,\pm}^{(s_j)}(\xi_j) \left[\prod_{k=1}^{j} t^{(s_k)}(\xi_k)^{-1}\right]$$

$$\Lambda_{z,\pm}^{(s)}(u) := \mathrm{Tr}_0 \left[S_0^{z,\pm} T_{0;1\ldots N}^{(s)}(u)\right] ; t^{(s)}(u) := \mathrm{Tr}_0 \left[T_{0;1\ldots N}^{(s)}(u)\right]$$

$$\Lambda_{z,\pm}^{(s)}(u) = \sum_{k=1}^{2s} t^{(s-\frac{k}{2})}(u_k)\mathcal{O}_{z,\pm}(u_{2k} - \alpha)t^{(\frac{k-1}{2})}(u_k - \alpha)$$

$$u_k = u + \frac{k\eta}{2} ; \alpha = \frac{2s+1}{2} ; \mathcal{O}_{z,\pm}(u) = \begin{cases} \frac{(A-D)(u)}{2} \\ C(u) \\ B(u) \end{cases}$$

# IV. FUSION

- A further important ingredient is the fusion procedure for quantum spin chains: a procedure for constructing higher spin objects (R-matrices, monodromy matrices ...) in terms of lower spin quantities (Clesbsch-Gordan decomposition).

P.P. Kulish, N. Yu. Reshetikhin and E.K. Sklyanin (1981) [fusion XXX chains]; A.N. Kirillov and N. Yu. Reshetikhin (1987) [fusion XXZ chains]; V.G. Drinfeld (1988) [quantum group approach]

- Several kinds of iterative relations follow from fusion:

$$t^{(s)}(x^+) = t^{(\frac{1}{2})}(x+s\eta)t^{(s-\frac{1}{2})}(x) - \chi(x^- + s\eta)t^{(s-1)}(x^-)$$

$$\chi(u) = A(u^+)D(u^-) - B(u^+)C(u^-) \quad \text{and} \quad u^\pm = u \pm \eta/2$$

- These iterative relations for the higher spin monodromy matrices imply similar relations for their eigenvalues on a Bethe state and allowed to find closed expressions for these eigenvalues

$$\Lambda^{(s)}(u,\{\lambda\}) = \sum_{\alpha=0}^{2s} C_\alpha^{(s)}(u) \prod_{p=1}^{\ell} \tau(\lambda_p,\alpha)$$

$$\tau(\lambda,\alpha) = \frac{\varphi(u^+ - \lambda + s\eta)\varphi(u^- - \lambda - s\eta)}{\varphi(u^+ - \lambda + (\alpha - s)\eta)\varphi(u^- - \lambda + (\alpha - s)\eta)}$$

$$C_\alpha^{(s)}(u) = \prod_{k=\alpha}^{2s-1} d(u^+ + (k-s)\eta) \qquad \text{and} \qquad C_{2s}^{(s)}(u) = 1$$

# V. FORM FACTORS

- We would like to compute the form factors (for spin $s_j$)

$$\boxed{F_\ell^{z,\pm}(j,\{\mu\},\{\lambda\}) = \langle\psi(\{\mu\})|S_j^{z,\pm}|\psi(\{\lambda\})\rangle}$$

- There are two basic results we need to use: the action of operators $A, D$ on a Bethe state

$$A(x)\,|\Psi(\{\lambda\})\rangle = \underbrace{\prod_{k=1}^{\ell}\frac{\varphi(\lambda_k - x + \eta)}{\varphi(\lambda_k - x)}\,|\Psi(\{\lambda\})\rangle}_{\text{direct term}}$$

$$\underbrace{-\sum_{p=1}^{\ell}\frac{\varphi(\eta)}{\varphi(\lambda_p - x)}\prod_{k\neq p}\frac{\varphi(\lambda_k - \lambda_p + \eta)}{\varphi(\lambda_k - \lambda_p)}B(x)\prod_{k\neq p}B(\lambda_k)\,|0\rangle}_{\text{indirect term}}$$

(similarly for $D(x)$) and the expression of the scalar product of a Bethe state and an arbitrary state:

V.E. Korepin and A.G. Izergin (1982-1985); N. Slavnov (1988).

$$\boxed{\langle\psi(\{\mu\})|\psi(\{\lambda\})\rangle = \frac{\det H(\{\mu\},\{\lambda\})}{\prod_{i<j}\varphi(\lambda_i - \lambda_j)\varphi(\mu_j - \mu_i)}}$$

$$H_{ab} = \frac{\varphi(\eta)}{\varphi(\lambda_a - \mu_b)}\left[\prod_{i\neq a}\varphi(\lambda_i - \mu_b + \eta) - d(\mu_b)\prod_{i\neq a}\varphi(\lambda_i - \mu_b - \eta)\right]$$

• Employing these formulae and the reconstruction of $S_j^{z,\pm}$ in terms of $\{A, B, C, D\}$ we have found

$$\boxed{\begin{array}{l} F_\ell^z(j,\{\mu\},\{\lambda\}) = \dfrac{\phi_{j-1}(\{\mu\})}{\phi_{j-1}(\{\lambda\})} \dfrac{s_j \det H - \sum\limits_{p=1}^{\ell} \det \mathcal{Z}^{(p)}(\xi_j)}{\prod\limits_{i<j} \varphi(\lambda_i - \lambda_j)\varphi(\mu_j - \mu_i)} \\ \phi_{j-1}(\{\mu\}) = \prod\limits_{k=1}^{j-1} \Lambda^{(s_k)}(\xi_k,\{\mu\}); \qquad \xi_j^{\pm} = \xi_j \pm \eta/2 \end{array}}$$

$$\mathcal{Z}^{(p)}(\xi_j)_{ab} = H_{ab} \qquad \text{for} \qquad b \neq p$$

$$\mathcal{Z}^{(p)}(\xi_j)_{ap} = \prod_{k=1}^{\ell} \frac{\varphi(\mu_k - \mu_p + \eta)\varphi(\mu_k - \xi_j^- - s_j\eta)}{\varphi(\lambda_k - \xi_j^- - s_j\eta)} \times \frac{\varphi(2\eta s_j)}{\varphi(\mu_a - \xi_j^- - s_j\eta)\varphi(\mu_a - \xi_j^- + s_j\eta)}$$

• This is a closed formula for all non-vanishing form factors of $S_j^z$ in an arbitrary spin $s_j$ representation.

• It holds for both XXX and XXZ chains, irrespectively of the spin representations sitting at other sites of the chain (those only enter the function $\phi_{j-1}$).

• As a consistency check, the total magnetization of the chain can be computed employing the formula above

$$\boxed{\mu = \sum_{j=1}^{N} \frac{F_\ell^z(j,\{\lambda\},\{\lambda\})}{\langle\psi(\{\lambda\})|\psi(\{\lambda\})\rangle} = \sum_{j=1}^{N} s_j - \ell}$$

$$F_\ell^+(j,\{\lambda\},\{\mu\}) = \frac{\phi_{j-1}(\{\lambda\})}{\phi_{j-1}(\{\mu\})} \frac{\prod_{k=1}^{\ell+1} \varphi(\mu_k - \xi_j^- + s_j\eta)}{\prod_{k=1}^{\ell} \varphi(\lambda_k - \xi_j^- + s_j\eta)} \times \frac{\det \mathcal{C}(\xi_j)}{\prod_{i<j} \varphi(\lambda_i - \lambda_j)\varphi(\mu_j - \mu_i)}$$

$$\mathcal{C}_{ab}(\xi_j) = \begin{cases} H_{ab} & \text{for} \quad b \neq \ell+1 \\ \dfrac{\varphi(2\eta s_j)}{\varphi(\mu_a - \xi_j^- - s_j\eta)\varphi(\mu_a - \xi_j^- + s_j\eta)} & \text{for} \quad b = \ell+1 \end{cases}$$

$$F_\ell^-(j,\{\mu\},\{\lambda\}) = \frac{\phi_j(\{\mu\})\phi_{j-1}(\{\mu\})}{\phi_{j-1}(\{\lambda\})\phi_j(\{\lambda\})} F_\ell^+(j,\{\mu\},\{\lambda\})$$

- This is a closed formula for all non-vanishing form factors of $S_j^\pm$ in an arbitrary spin $s_j$ representation.
- It holds for both XXX and XXZ chains, irrespectively of the spin representations sitting at other sites of the chain (those only enter the function $\phi_{j-1}$).
- In order to obtain these formulae, highly non-trivial algebraic identities involving the functions $\Lambda^{(s)}(u,\{\lambda\})$ needed to be proven.

# VI. CONCLUSIONS

- The Algebraic Bethe ansatz technique, together with the solution of the inverse scattering problem can be successfully employed to compute form factors of spin operators for higher spins and mixed spin chains.
- These results apply both to XXX and XXZ spin chains. In the latter case one must mention that the spin operators considered are still the generators of $su(2)$.
- Our results can be used for the study of specially interesting models, such as impurity systems and alternating chains, two kinds of systems whose thermodynamic properties have been extensively studied in the BA framework.
- Finally, we expect these results to be useful for numerical computations. For the spin 1/2 case it has been proven (J.-S. Caux, R. Hagemans and J.-M. Maillet (2005)) that the numerical evaluation of dynamical correlation functions, by employing the form factors obtained by this approach, leads to very precise results which can be matched with experimental measurements.