Solutions to Dynamical Systems Coursework 2

Optional Coursework

1. *Consider the following first order differential equation*

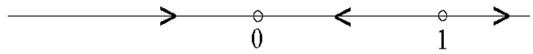
$$\frac{dy}{dt} = \dot{y} = y(y-1),$$

(a) *Find and classify the fixed points of the equation.*
The equation has fixed points at $X(y) = y(y-1) = 0$, that is $y = 0, 1$. In order to classify the fixed points we compute:

$$X'(y) = 2y - 1,$$

and evaluate $X'(0) = -1 < 0$ and $X'(1) = 1 > 0$. Therefore $y = 0$ is an attractor and $y = 1$ is a repellor.

(b) *Draw the phase space diagram associated to the equation.*
The phase diagram can be easily drawn once the fixed points have been classified. It would be as shown below:



(c) *Find the solution to the equation for the following initial conditions (there will be a different solution for each case!):*

- $y(0) = -1$,
- $y(0) = \frac{1}{2}$,
- $y(0) = 2$.

(d) *For each solution, indicate the range of values of t for which it is defined. Indicate the behaviour of each solution for $t \to \pm\infty$.*
Let us try to find the general solution first and then the particular solutions. We can solve the equation above by separation of variables:

$$\int_{y_0}^{y} \frac{dy}{y(y-1)} = \int_{t_0}^{t} dt = t - t_0.$$

The y-integral can be computed as:

$$\int_{y_0}^{y} \frac{dy}{y(y-1)} = \int_{y_0}^{y} \left(-\frac{1}{y} + \frac{1}{y-1}\right) dy = \log\left(\frac{y_0(y-1)}{y(y_0-1)}\right).$$

Taking exponential on both sides of the equation,

$$e^{t-t_0} = \frac{y_0(y-1)}{y(y_0-1)}, \quad \Rightarrow \quad \left(\frac{y_0-1}{y_0}\right) e^{t-t_0} = \frac{y-1}{y}, \quad \Rightarrow \quad y(t) = \frac{1}{1 - \frac{y_0-1}{y_0} e^{t-t_0}} = \frac{y_0}{y_0 - (y_0-1)e^{t-t_0}}.$$

For the initial condition $y(0) = -1$ we have that $y_0 = -1$ and $t_0 = 0$ so, substituting we obtain:

$$y(t) = \frac{-1}{-1 + 2e^t}.$$

This solution is well-defined whenever the denominator is not vanishing. The denominator vanishes at $-1 + 2e^t = 0$ which gives $t = -\log(2) = -0.693147$. So for this value of t our solution will diverge. The interval of definition of $y(t)$ must exclude the point $t = -\ln(2)$ but must include the initial condition at $t_0 = 0 > -\ln(2)$. Therefore $y(t)$ is only defined for $t \in (-\ln(2), \infty)$.
We can also see that when $t \to \infty$ the term e^t in the denominator becomes very large, so that the function $y(t)$ tends to the value zero when $t \to \infty$. This is what you would expect from the phase diagram, as we saw before that $y = 0$ is an attractor, so all trajectories near $y = 0$ tend to approach the value 0 as t increases.
For the initial condition $y(0) = \frac{1}{2}$ we have $y_0 = \frac{1}{2}$ and $t_0 = 0$ and therefore,

$$y(t) = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2}e^t} = \frac{1}{1 + e^t}.$$

In this case the denominator is never zero, so this function is well-defined for all values of t, that is $t \in (-\infty, \infty)$.
As before, when $t \to \infty$, the function e^t becomes very large so that $y \to 0$. This is again the expected behaviour from the phase diagram, as $y = 0$ is an attractor. On the other hand, when $t \to -\infty$ the function $e^t \to 0$ so that $y \to 1$. This is also the expected behaviour, as $y = 1$ is a repellor. Trajectories tend to approach the fixed point $y = 1$ as $t \to -\infty$.
Finally, for the initial condition $y(0) = 2$ we obtain the solution:

$$y(t) = \frac{2}{2 - e^t}.$$

In this case we have a similar situation as for the first initial condition, that is the denominator can vanish. It vanishes for the value $t = \ln(2) = 0.693147$. The interval of definition of $y(t)$ must exclude the point $t = \ln(2)$ but must include the initial condition at $t_0 = 0 < \ln(2)$. Therefore $y(t)$ is only defined for $t \in (-\infty, \ln(2))$. As $t \to -\infty$ we find the expected behaviour $y \to 1$.

(e) *Sketch the solutions that you just obtained in the same diagram, drawing also the lines corresponding to the fixed points.*

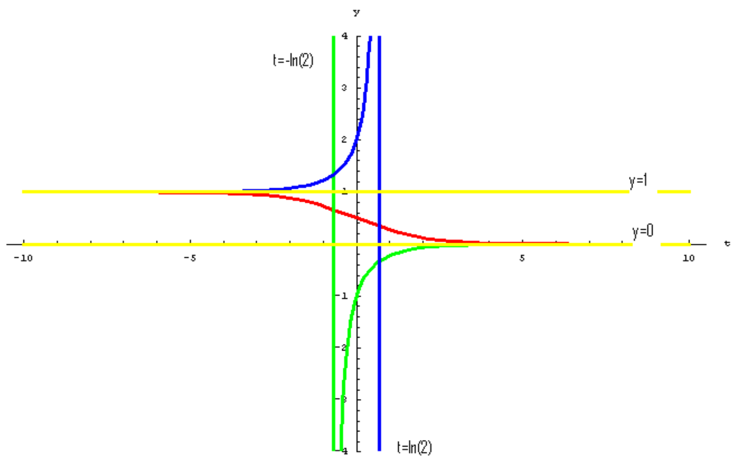


Figure 1: The yellow lines represent the fixed points $y = 0$ and $y = 1$. The blue curve is the solution with initial condition $y(0) = 2$. It diverges at the vertical line $t = \ln(2)$ (also in blue). The green curve is the solution with initial condition $y(0) = -1$. It diverges at the vertical line $t = -\ln(2)$ (also in green). Finally, the curve in red represents the solution with initial condition $y(0) = \frac{1}{2}$ which, as predicted, approaches the fixed point $y = 0$ for large t and the fixed point $y = 1$ for small t.

(f) *Linearize the original equation about the fixed points. Solve the resulting linear equation for the initial condition $y(0) = \frac{1}{2}$.*
In order to linearize the equation, we need to Taylor expand the function $X(y) = y(y-1)$ about each of the fixed points. Let $y = a$ be a fixed point. Then Taylor expanding we get

$$X(y) \approx X(a) + X'(a)(y - a) + \ldots$$

since a is a fixed point $X(a) = 0$ so that the equation becomes simply

$$X(y) \approx X'(a)(y - a),$$

or, introducing the variable $z = y - a$,

$$X(z) = \frac{dz}{dt} = X'(a)z,$$

which has solution $z = Ae^{X'(a)t}$, with A and integration constant. In our case $X'(a) = 2a - 1$ so that the linearized solution would be

$$y = a + Ae^{(2a-1)t}.$$

For $a = 0$ this gives $y = Ae^{-t}$. The initial condition $y(0) = 1/2$ fixes $A = 1/2$.
On the other hand for $a = 1$ we get $y = 1 + Ae^{t}$ and the initial condition now fixes $A = -1/2$.
So, the linearized solutions about the fixed points are

$$y = \frac{e^{-t}}{2}, \quad \text{and} \quad y = 1 - \frac{e^{t}}{2}.$$

2. Consider the following second order linear differential equation:

$$\ddot{y} - 2\dot{y} + 2y = 1,$$

(a) *Use the methods that you learned in first year calculus to find the general solution to this equation.*
We start by solving the homogeneous equation $\ddot{y} - 2\dot{y} + 2y = 0$. As usual we try solutions of the form $y = e^{mt}$ for some m that will be fixed by the equation. Substituting into the equation we get the condition

$$m^2 - 2m + 2 = 0,$$

which is solved by $m = 1 \pm i$. This means that the general solution of the homogeneous equation is of the form $Ae^{t+it} + Be^{t-it}$, for some constants A and B. It is generally preferable to express this solution in terms of real functions, so instead of having $e^{\pm it}$ we can write $C_1 e^t \cos(t) + C_2 e^t \sin(t)$ for some new arbitrary constants C_1 and C_2. This is true because $e^{\pm it} = \cos(t) \pm i \sin(t)$. So, the new constants C_1 and C_2 are related to the old ones as $C_1 = A + B$ and $C_2 = i(A - B)$.
Once we have the general solution to the homogeneous equation we just need to find one particular solution of the inhomogeneous one. In this case such solution is simply the constant $y = \frac{1}{2}$ (we can easily see that this solves the equation).
Thus the general solution to the equation is

$$y(t) = C_1 e^t \cos(t) + C_2 e^t \sin(t) + \frac{1}{2}.$$

(b) *We saw in the lecture that an equation of this type can always be transformed into two first order linear differential equations by redefining $y = x_1$ and $\dot{y} = x_2$. Write down the new equations in terms of the variables x_1 and x_2.*
The equations become

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = 2x_2 - 2x_1 + 1.$$

(c) *Write the equations of section (b) in the matrix form $\dot{\underline{x}} = A\underline{x} + \underline{b}$ which we have been using in the lecture.*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(d) *Find the fixed point of the system of equations, $\underline{a}$. Hence find $\underline{z}$ so that the original equation can be brought into the form $\dot{\underline{z}} = A\underline{z}$.*
The fixed point is the solution to $A\underline{a} + \underline{b} = 0$, that is $\underline{a} = -A^{-1}\underline{b}$, so in this case it is the point

$$\underline{a} = -\frac{1}{2} \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}.$$

We can therefore define $\underline{z} = \underline{x} - \underline{a}$. Substituting in the original equation

$$\dot{\underline{x}} = \dot{\underline{z}} = A\underline{x} + \underline{b} = A(\underline{z} + \underline{a}) + \underline{b} = A\underline{z} + \underbrace{A\underline{a} + \underline{b}}_{=0} = A\underline{z}.$$

(e) *Find the eigenvalues and eigenvectors of the matrix A.*
The eigenvalues are the zeroes of the characteristic polynomial

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda)+2 = \lambda^2-2\lambda+2=0 \qquad \Rightarrow \quad \lambda_1 = 1+i, \quad \lambda_2 = 1-i.$$

(f) *Find the Jordan Normal form of A. Construct the matrix P that relates the two matrices as $A = PJP^{-1}$.*
The Jordan Normal form is

$$J = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The matrix P can be obtained by computing the eigenvectors of A. In this case, the eigenvectors are complex conjugated to each other (as the eigenvalues are!). Therefore, it is enough finding one of the eigenvectors. We solve

$$\begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (1+i)\begin{pmatrix} a \\ b \end{pmatrix}.$$

This gives equations

$$b = (1+i)a, \qquad -2a+2b = (1+i)b,$$

or

$$b = (1+i)a, \qquad -2a = (-1+i)b.$$

Although they look different at first sight, this two equations are actually identical (you just need to multiply both sides of the second equation by $(1+i)$ and you get the first one. Therefore the equations are not independent, which means we can fix one of the constants to whichever value we like (as long as the eigenvector is not zero) and then solve for the other constant. For example, let us take $a = 1$, then $b = 1+i$. Therefore, the eigenvectors are

$$\underline{E}_1 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}, \qquad \underline{E}_2 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}.$$

The matrix P is given in terms of the real and imaginary parts of $\underline{E}_1$,

$$\underline{e}_1 = \mathrm{Re}(\underline{E}_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \underline{e}_2 = \mathrm{Im}(\underline{E}_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$P = (\underline{e}_1, \underline{e}_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Computing

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = J.$$

(g) *Find the general solution to the system of equations $\dot{\underline{y}} = J\underline{y}$.*
As usual for this kind of Jordan matrix, one finds that, by changing variables to $y_1 = r\cos(t)$ and $y_2 = r\sin(t)$, the equations become

$$\dot{r} = r, \qquad \dot{\theta} = -1,$$

so that $r = r_0 e^t$ and $\theta = -t+\theta_0$ and the solutions become,

$$y_1 = r_0 e^t \cos(-t+\theta_0), \qquad y_2 = r_0 e^t \sin(-t+\theta_0).$$

(h) *Using the solution found in part (g) find the solution for $\underline{z}$ and then for $\underline{x}$. Check that this solution is actually the same you found in section (a).*

$$\underline{z} = P\underline{y} = r_0 e^t\cos(-t+\theta_0)P\begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0 e^t \sin(-t+\theta_0)P\begin{pmatrix} 0 \\ 1 \end{pmatrix} = r_0 e^t\cos(-t+\theta_0)\underline{e}_1 + r_0 e^t\sin(-t+\theta_0)\underline{e}_2.$$

$$
\begin{aligned}
\underline{x} = \underline{z} + \underline{a} &= r_0 e^t \cos(-t+\theta_0)\underline{e}_1 + r_0 e^t \sin(-t+\theta_0)\underline{e}_2 + \underline{a} \\
&= r_0 e^t \cos(-t+\theta_0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + r_0 e^t \sin(-t+\theta_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}.
\end{aligned}
$$

This gives

$$
x_1 = r_0 e^t \cos(-t+\theta_0) + \frac{1}{2}, \qquad x_2 = r_0 e^t \cos(-t+\theta_0) + r_0 e^t \sin(-t+\theta_0).
$$

Recall from section (b) that $y = x_1$ in the original equation. So x_1 should be equivalent to the solution we found in section (a). Although it does not look the same, it is easy to show that it is equivalent by expanding the cosine function above

$$
\cos(-t+\theta_0) = \cos(t)\cos(\theta_0) + \sin(t)\sin(\theta_0),
$$

and redefining

$$
r_0 \cos(\theta_0) = C_1, \qquad r_0 \sin(\theta_0) = C_2,
$$

we recover exactly the solution of part (a). From (b) we also have that $x_2 = \dot{x}_1$ and we can see directly that this is the case.

(i) *Classify the fixed point at the origin and draw the phase space diagram both in the $y_1 - y_2$ and in the $x_1 - x_2$ planes.*
Since the eigenvalues are complex conjugated to each other and their real part is positive, the fixed point is an unstable focus. The phase diagrams are presented below.

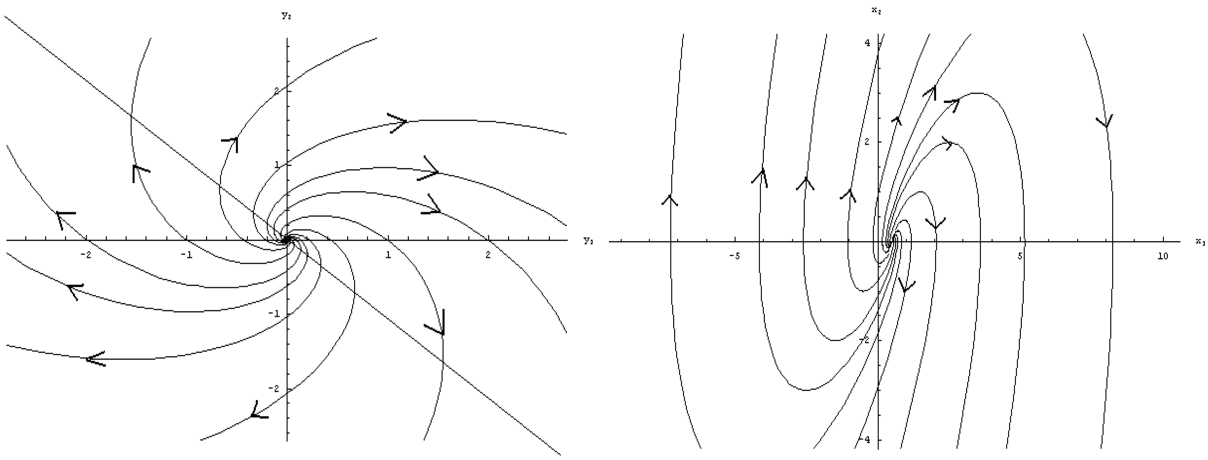


Figure 2: As expected the two phase diagrams are a slight distortion of each other. The most obvious change is that the fixed point changes from being at the origin to being at $(\frac{1}{2}, 0)$. Because the real part of the eigevalues is positive this produces the exponentials e^t that multiply the solutions and make x_1, x_2, y_1, y_2 grow exponentially as $t \to \infty$. The factor that determines the change in shape of the trajectories between the two diagrams is the position at which trajectories become exactly vertical (the slope becomes infinite). In the second diagram $x_1 - x_2$ one can see directly that trajectories are perpendicular to the x_1 axis. This means that $\frac{dx_2}{dx_1} \to \infty$ at $x_2 = 0$ and this follows directly from the equations we wrote in section (b). On the other hand, in the $y_1 - y_2$ diagram, trajectories are not perpendicular to any of the coordinate axis. In fact, $\frac{dy_2}{dy_1} \to \infty$ when $y_1 = -y_2$ (the diagonal line in the first picture above). This also follows from looking at the equations in the variables y_1, y_2 which we found before ($\dot{y}_1 = y_1 + y_2$ and $\dot{y}_2 = -y_1 + y_2$).