Attempt three of the following five questions in 2 hours

1. Consider the following first order differential equation:

$$\frac{dy}{dt} = -y(y^2 - 4). \tag{1}$$

(a) Find and classify all the fixed points of (1). Hence draw the phase diagram associated to (1).

(b) Solve (1) for each of the following initial conditions:

- $y(0) = 3$,
- $y(0) = -1$,

indicating the range of values of t for which each solution is defined.

(c) Provide a sketch of your solutions against the variable t, clearly showing the behaviour of the various functions as $t \to \pm\infty$.

(d) Write down the linearised version of equation (1) about a generic fixed point $y = a$.

(e) Use the result of (d) to find the solution of the linearised equation about the smallest of the fixed points of (1), if the initial condition is $y(0) = -3$.

2. Consider the following system of linear differential equations:

$$\frac{dx_1}{dt} = \dot{x}_1 = \alpha x_1 + \gamma x_2 + c_1, \qquad \frac{dx_2}{dt} = \dot{x}_2 = \beta x_1 + \alpha x_2 + c_2, \tag{2}$$

where $\alpha, \beta, \gamma, c_1, c_2$ are real constants.

(a) Write the equations in the standard form $\dot{\underline{x}} = A\underline{x} + \underline{b}$. Write down the vector $\underline{a}$ which is a fixed point of (2).

(b) Compute the eigenvalues and eigenvectors of the matrix A.

(c) Define the vector $\underline{z}$ in terms of which the equation can be re-written in the form $\dot{\underline{z}} = A\underline{z}$.

(d) Indicate the conditions (if any) that α, β and γ must satisfy so that the fixed point is:

- an improper node,
- a focus,
- a star node.

(e) If $\alpha = 1$, $\beta = -2$ and $\gamma = 2$ classify the fixed point. Find the values of c_1 and c_2 so that the fixed point is $\underline{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(f) For $\alpha = 1$, $\beta = -2$ and $\gamma = 2$ show that the matrix A is already in its Jordan Normal form. For these values of α, β, γ and the values of c_1, c_2 obtained in (e) find the general solution of equations (2).

3. Consider the following system of first order nonlinear differential equations:

$$\dot{x}_1 = x_1(1 - x_1 - x_2), \qquad \dot{x}_2 = x_2(1 - x_1 - x_2). \tag{3}$$

(a) Show that the fixed points of the system are either at the origin or on a line.

(b) Calculate the Jacobian matrix for a general point (x_1, x_2) and deduce that the fixed point at the origin is simple, whereas the fixed points on the line are not.

(c) By writing down the linearisation about the fixed point at the origin classify its nature. Solve the linearised equation.

(d) By dividing the two equations in (3) obtain a new equation of the form $\frac{dx_1}{dx_2} = X(x_1, x_2)$. Find the general solution to this equation. Explain why your solution is what one would expect given the nature of the fixed point at the origin.

(e) Use the results of (c) and (d) to sketch the phase space diagram associated to (3). Pay special attention to the behaviour of phase space trajectories near the line of fixed points found in (a). Does the direction of the arrows change at this line? Why?

4. Consider the following second order linear differential equation:

$$\ddot{x} - 4\dot{x} - 5x = 0, \tag{4}$$

where $\ddot{x} = \frac{d^2x}{dt^2}$ and $\dot{x} = \frac{dx}{dt}$.

(a) Consider now the following two first order linear differential equations,

$$\dot{x}_1 = ax_1 + bx_2 + c, \qquad \dot{x}_2 = dx_1 + ex_2 + f, \tag{5}$$

where $x_1 = x$ above. Find the values of the constants a, b, c, d, e and f so that the system (5) is equivalent to (4).

(b) Using the result of (a) write equation (4) in the standard matrix form $\dot{\underline{x}} = A\underline{x} + \underline{b}$.

(c) Find the eigenvalues and eigenvectors of the matrix A. Hence construct the Jordan normal form of A and the matrix P which relates A to its Jordan form J as $A = PJP^{-1}$.

(d) Find the fixed point of the system of equations and classify its nature.

(e) Find the general solution of the system of equations $\dot{\underline{y}} = J\underline{y}$. Hence find the general solution for $\underline{x}$.

(f) Sketch the phase diagram both in the $y_1 - y_2$ plane and in the $x_1 - x_2$ plane.

Turn over...

5. Consider the following first order nonlinear differential equations

$$\dot{x}_1 = (1 - \frac{x_1}{2} - x_2)x_1, \qquad \dot{x}_2 = -(1 + \frac{x_2}{2} - x_1)x_2, \tag{6}$$

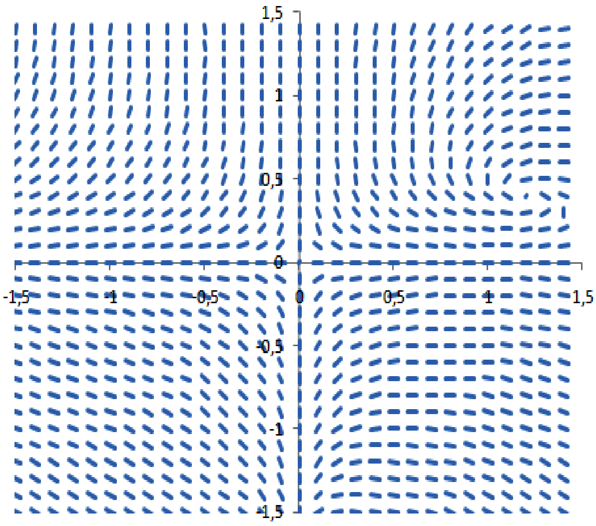
which can be interpreted as modelling the populations of two competing species.

(a) Find the four fixed points of the system.

(b) Find the general form of the Jacobian matrix of the system and evaluate it at each fixed point. Classify the fixed points according to their linearisation.

(c) For only those fixed points lying in the region $-1.5 \leq x_1 \leq 1.5$ and $-1.5 \leq x_2 \leq 1.5$ solve the linearised equations.

(d) In the same region of the $x_1 - x_2$ plane considered in section (c) and using the solutions to the linearised equations, sketch the phase diagram associated to the nonlinear system of equations. You may employ the vector field diagram below as a guiding tool in order to produce your sketch.



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