MA2605 DYNAMICAL SYSTEMS 2011 EXAM'S SOLUTIONS

1. (a) The equation is non-linear and autonomous. It is non-linear because in general a linear combination of solutions is not a solution to the equation. It is autonomous because the variable t does not explicitly appear in the r.h.s. of the equation. **[2]**

(b) The fixed points are the zeroes of the $\cos(y)$ function, that is

$$y = \frac{\pi}{2} + n\pi, \qquad \text{with} \quad n = 0, \pm 1, \pm 2 \dots$$

Concerning the classification of the fixed points, students could do this by either studying the sign of $\cos(y)$ in the various regions or by using the linearisation method.

Using the first method, we have that $\cos(y) > 0$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and therefore, due to periodicity, it is positive in all intervals of the form $-\frac{\pi}{2} + 2n\pi < y < \frac{\pi}{2} + 2n\pi$ for $n = 0, \pm 1, \pm 2 \dots$. Similarly, $\cos(y) < 0$ in the remaining regions, that is $\frac{\pi}{2} + 2n\pi < y < \frac{3\pi}{2} + 2n\pi$ with $n = 0, \pm 1, \pm 2 \dots$. This implies that the fixed points of the form $\frac{\pi}{2} + 2\pi n$, with $n = 0, \pm 1, \pm 2 \dots$ are attractors and the fixed points $\frac{3\pi}{2} + 2\pi n$ with $n = 0, \pm 1, \pm 2 \dots$ are repellors.

Using the linearisation method one looks at the derivative of $\cos(y)$, that is $-\sin(y)$ and studies its sign at each fixed point. We find that $-\sin\left(\frac{\pi}{2} + 2\pi n\right) = -1$ whereas $-\sin\left(\frac{3\pi}{2} + 2\pi n\right) = 1$. This immediately implies the same classification as above. **[6]**

(c)

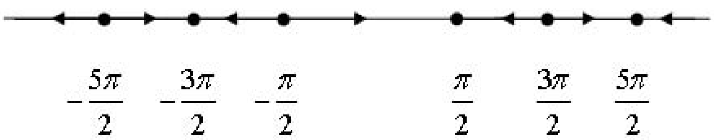

Figure 1: Phase space diagram of (1)

[3]

(d) Linearisation means replacing $X(y)$ by its linear approximation near the fixed point. That is Taylor expanding $X(y) \approx X(a) + X'(a)(y - a) + \cdots$ and using the fact that a is a fixed point, that is $X(a) = 0$. This gives equation (3). **[3]**

(e) The general solution to equation (3) is $z = Ae^{X'(a)t}$ where A is an arbitrary constant which will be fixed by the initial conditions. For y this implies $y = a + Ae^{X'(a)t}$. If we want that $y = y_0$ for $t = t_0$ then we have to choose $A = (y_0 - a)e^{-X'(a)t_0}$ which gives $y = a + (y_0 - a)e^{X'(a)(t - t_0)}$. **[3]**

(f) Adapting the result of (e) to our equation we have that $X(y) = \cos(y)$, $X'(y) = -\sin(y)$. At $-\frac{\pi}{2}$, $X'(-\frac{\pi}{2}) = 1$ so that the linearised solution would be $y = -\frac{\pi}{2} + (1 + \frac{\pi}{2})e^{t-1}$. **[3]**

(g) We can solve the equation by separation of variables:

$$\int_0^y \frac{dy}{\cos(y)} = \int_1^t dt = t - 1.$$

Using the change of variables suggested we have that $x = \tan(y/2)$ so that $dx = \frac{1}{2\cos^2(y/2)}dy = \frac{1+x^2}{2}dy$. Substituting in the first integral and simplifying we find,

$$\int_0^y \frac{dy}{\cos(y)} = \int_0^x \frac{2dx}{1-x^2} = \int_0^x \frac{dx}{1-x} + \int_0^x \frac{dx}{1+x} = \log\left(\frac{1+x}{1-x}\right).$$

Therefore we have that

$$\log\left(\frac{1+x}{1-x}\right) = t-1 \quad \Rightarrow \quad \frac{1+x}{1-x} = e^{t-1} \quad \Rightarrow \quad x = \tan(y/2) = \frac{e^{t-1}-1}{e^{t-1}+1}.$$

Therefore

$$y = 2\tan^{-1}\left(\frac{e^{t-1}-1}{e^{t-1}+1}\right).$$

The argument of the function $\tan^{-1}$ can be any real positive or negative number. There could be a problem if the denominator $e^{t-1}+1$ would vanish, but this can never happen for finite real values of t. Therefore the solution is defined for $t \in I = (-\infty, \infty)$. **[5]**

2. (a)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

[1]

(b) The fixed point is the solution of the equation $A\underline{a} + \underline{b} = 0$, that is $\underline{a} = -A^{-1}\underline{b}$. The fixed point is simple because $\det(A) = 13 \neq 0$. In order to find the fixed point we need to evaluate the inverse of A:

$$A^{-1} = \frac{1}{13}\begin{pmatrix} 4 & 5 \\ -1 & 2 \end{pmatrix},$$

and multiply by $\underline{b}$,

$$\underline{a} = -\frac{1}{13}\begin{pmatrix} 4 & 5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{13}\begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

[3]

(c) The eigenvalues of A are obtained by solving

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -5 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) + 5 = \lambda^2 - 6\lambda + 13 = 0,$$

which gives solutions $\lambda_1 = 3 + 2i$ and $\lambda_2 = 3 - 2i$. The eigenvalues are complex conjugated to each other with non-vanishing positive real part, which means that the fixed point is an <u>unstable focus</u>.

We now just have to compute one eigenvector, as the other one would simply be its complex conjugate. We solve,

$$\begin{pmatrix} 2 & -5 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (3+2i)\begin{pmatrix} a \\ b \end{pmatrix},$$

which gives equations $2a - 5b = (3+2i)a$ and $a + 4b = (3+2i)b$. The two equations are in fact equivalent. We can see that by solving the first equation for b to $b = -(1+2i)a/5$ and doing the same for the second equation $b = a/(2i-1)$. The solutions look different by if we rewrite the last equation as $b = \frac{a(2i+1)}{(2i-1)(2i+1)} = -\frac{a(2i+1)}{5}$ they are in fact the same. This means that we can take any values of a, b that solve this one equation. For example $a = 5$ and $b = -(1 + 2i)$. Therefore, the eigenvectors are

$$\underline{E}_1 = \begin{pmatrix} 5 \\ -1 - 2i \end{pmatrix}, \qquad \underline{E}_2 = \begin{pmatrix} 5 \\ -1 + 2i \end{pmatrix}.$$

[5]

(d) The vector is just $\underline{z} = \underline{x} - \underline{a}$. **[1]**

(e) The Jordan normal form is

$$J = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix},$$

and the matrix P is $P = (\Re(\underline{E}_1), \Im(\underline{E}_1))$, that is

$$P = \begin{pmatrix} 5 & 0 \\ -1 & -2 \end{pmatrix}.$$

The relationship between vectors is $\underline{z} = P\underline{y}$. **[5]**

(f) The general solution for this kind of fixed point is always

$$\underline{y} = r_0 e^{3t} \cos(-2t + \theta_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0 e^{3t} \sin(-2t + \theta_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

and the solution for $\underline{x}$ is just $\underline{z}$ plus the fixed point, that is

$$\underline{x} = -\frac{1}{13} \begin{pmatrix} 4 \\ -1 \end{pmatrix} + r_0 e^{3t} \cos(-2t + \theta_0) \begin{pmatrix} 5 \\ -1 \end{pmatrix} + r_0 e^{3t} \sin(-2t + \theta_0) \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

[5]

(g) See below: **[5]**

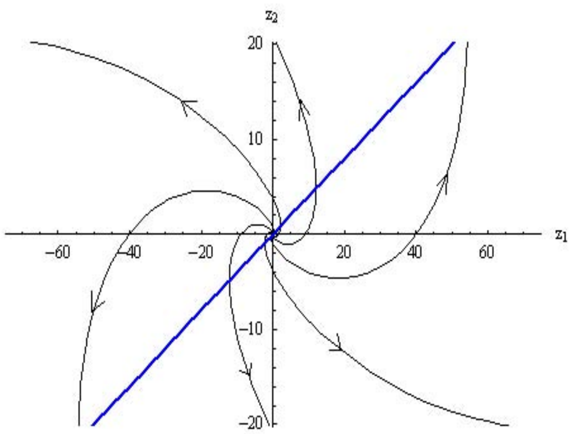

Figure 2: Here the arrows point away from the origin as the fixed point is unstable. The other key feature is the blue line which is the line $2z_1 - 5z_2 = 0$. At this line we have from the original equation that $\dot{z}_1 = 0$, which means that the trajectories have infinite slope when they meet the line (they are vertical).

3. (a) The fixed points are the solutions of

$$X_1(x_1,x_2) = 2 - x_1 - x_2 \quad \text{and} \quad X_2(x_1,x_2) = 2x_2^2 e^{x_1}(2 - x_1 - x_2) = 0,$$

clearly one solution to this is $x_1 + x_2 = 2$ which is the equation of a line in the $x_1 - x_2$ plane. The second equation is also solved by $x_2 = 0$, but this does not solve the first equation, so it does not correspond to any extra fixed points. **[3]**

(b) The Jacobian matrix is

$$A_{(x_1,x_2)} = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{pmatrix}_{(x_1,x_2)} = \begin{pmatrix} -1 & -1 \\ 2x_2^2 e^{x_1}(1 - x_1 - x_2) & 2x_2 e^{x_1}(4 - 3x_2 - 2x_1) \end{pmatrix}_{(x_1,x_2)}$$

[5]

(c) If we insert the condition $x_1 + x_2 = 2$ into the Jacobian matrix above we find

$$A_{(x_1,2-x_1)} = \begin{pmatrix} -1 & -1 \\ -2(2-x_1)^2 e^{x_1} & -2(2-x_1)^2 e^{x_1} \end{pmatrix}_{(x_1,2-x_1)}.$$

The determinant of this matrix is clearly zero, therefore the fixed points on the line are not simple. **[5]**

(d) The new equation is

$$\frac{dx_2}{dx_1} = 2x_2^2 e^{x_1},$$

this can be solved using separation of variables

$$\frac{dx_2}{2x_2^2} = e^{x_1} dx_1 \quad \Rightarrow \quad -\frac{1}{2x_2} = e^{x_1} + C$$

which gives solution

$$x_2 = -\frac{1}{2(e^{x_1} + C)},$$

where C is an arbitrary constant. The particular solution that satisfies the condition $x_2 = -\frac{1}{4}$ for $x_1 = 0$ is obtained by solving $-\frac{1}{4} = -\frac{1}{2(1+C)}$, which fixes $C = 1$. **[6]**

(e) Since $x_2 = -\frac{1}{2(e^{x_1}+1)}$ we see that when $x_1 \to \infty$, $x_2 \to 0$ and when $x_1 \to -\infty$, $x_2 \to -\frac{1}{2}$. These features allow for a fairly accurate sketch of the function, as seen in figure (3).

When the trajectory crosses the line $x_1 + x_2 = 2$ there should be a change in the direction of the arrows. This is because the sign of $\dot{x}_1$ changes as we move from the region $x_1 + x_2 < 2$ to the region $x_1 + x_2 > 2$. The first region is on the l.h.s. of the line and in there $\dot{x}_1 > 0$, whereas in the second region on the r.h.s. of the line $\dot{x}_1 < 0$. The behaviour of $\dot{x}_2$ is exactly the same as for $\dot{x}_1$ since the pre-factor $2x_2^2 e^{x_1}$ is always positive (the sign of of $\dot{x}_2$ only depends on the sign of $2 - x_1 - x_2$). **[6]**

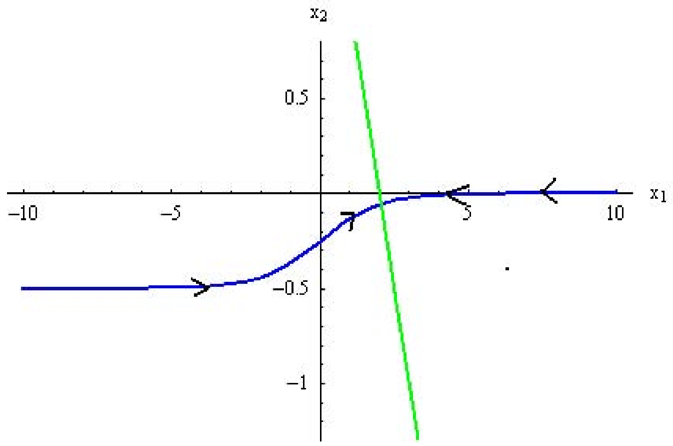

Figure 3: The blue line is the function $x_2 = -\frac{1}{2(x_1+1)}$ and the green line is $x_2 = 2 - x_1$.

4. (a) The usual change of variables is $x_1 = x$ and $x_2 = \dot{x}_1$. Then the equation becomes

$$\dot{x}_2 - 6x_2 + 8x_1 = 0, \qquad \dot{x}_1 = x_2.$$

[3]

(b) The equation is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

[2]

(c) The eigenvalues are the solutions to

$$\begin{vmatrix} -\lambda & 1 \\ -8 & 6-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 = 0,$$

which are $\lambda_1 = 4$ and $\lambda_2 = 2$. Thus the Jordan normal form of A is

$$J = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvectors are the solutions of

$$\begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 4 \begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix}.$$

The first equation gives $b = 4a$ and $-8a + 6b = 4b$. The two equations are equivalent, so we just choose one solution, such as $a = 1$, $b = 4$. Similarly, the second equation has a solution for $a = 1$ and $b = 2$. The eigenvectors are

$$\underline{E}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \qquad \underline{E}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

[5]

(d) Since the eigenvalues are both real, positive and different from each other the fixed point is an <u>unstable node</u>. **[3]**

(e) The general solution in the canonical coordinates is

$$\underline{y} = C_1 e^{4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where C_1, C_2 are arbitrary constants. The general solution on the original coordinates is

$$\underline{x} = C_1 e^{4t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

To satisfy the initial condition we need to solve

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which gives the conditions $1 = C_1 + C_2$ and $-1 = 4C_1 + 2C_2$. They are solved by $C_1 = -3/2$ and $C_2 = 5/2$. **[6]**

(f)

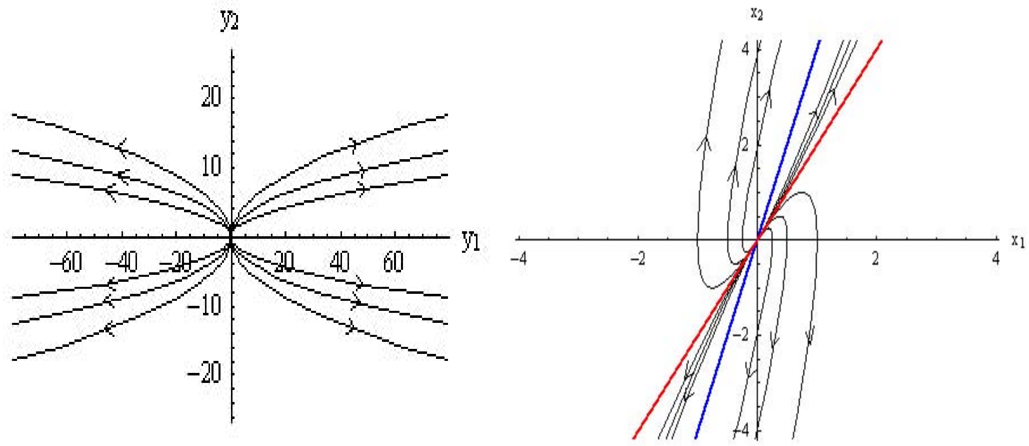

Figure 4: Phase diagrams in the $y_1 - y_2$ and $x_1 - x_2$ coordinates. In both cases the arrows point away from the origin, as the fixed point is unstable. In the second diagram the blue and red line represent the directions of the eigenvectors $\underline{E}_1$ and $\underline{E}_2$, respectively. The eigenvector $\underline{E}_1$ plays the same role as the y_1-axis in the first diagram, that is the trajectories tend to be parallel to the direction of $\underline{E}_1$ as $t \to \infty$. The eigenvector $\underline{E}_2$ takes over the role of the y_2-axis in the first diagram, so the trajectories tend to become flat along the direction of $\underline{E}_2$.

[6]

5. (a) The fixed points are the values of (x_1, x_2) for which the r.h.s. of both equations is zero. For the first equation there are two solutions $x_1 = \pm\frac{\pi}{2}$.
When substituting into the 2nd equation we find that for $x_1 = \frac{\pi}{2}$ need $\cos x_2 = 1$, which corresponds to $x_2 = 2n\pi$ with $n = 0, \pm1, \pm2 \ldots$. For $x_1 = -\frac{\pi}{2}$ need $\cos x_2 = -1$, which corresponds to $x_2 = (1+2n)\pi$ with $n = 0, \pm1, \pm2 \ldots$. In summary, there are infinitely many points of the form

$$\left(\frac{\pi}{2}, 2n\pi\right) \quad \text{and} \quad \left(-\frac{\pi}{2}, (1+2n)\pi\right) \quad \text{with} n = 0, \pm1, \pm2 \ldots$$

[6]

(b) The Jacobian matrix is

$$A_{(x_1,x_2)} = \begin{pmatrix} 2x_1 & 0 \\ \cos x_1 & \sin x_2 \end{pmatrix}_{(x_1,x_2)}.$$

The fixed points that have $0 < x_2 < 3\pi$ are $(\frac{\pi}{2}, 2\pi)$ and $(-\frac{\pi}{2}, \pi)$. The Jacobian matrix at those points is

$$A_{(\frac{\pi}{2},2\pi)} = \begin{pmatrix} \pi & 0 \\ 0 & 0 \end{pmatrix}_{(\frac{\pi}{2},2\pi)}, \qquad A_{(-\frac{\pi}{2},\pi)} = \begin{pmatrix} -\pi & 0 \\ 0 & 0 \end{pmatrix}_{(-\frac{\pi}{2},\pi)}$$

[6]

Many students rightly noticed that these two matrices are singular and therefore the remaining sections of the question could not have been answered. This was due to a typo on the original set of equations.

When marking the exam, the remaining 13 points that could have been gained for this question have been awarded to all students that did (a) and (b) and stopped because they could not do the remaining sections and also to students who attempted any of sections (c), (d) and (e) but could either not complete them or completed them incorrectly due to the typo.