Solutions to $\dot{\underline{x}} = A\underline{x}$

Let λ_1, λ_2 be the eigenvalues of A with eigenvectors $\underline{E}_1, \underline{E}_2$, then

Type I If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$:

$$A = PJP^{-1} \qquad \text{with} \qquad J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \text{and} \qquad P = (\underline{E}_1, \underline{E}_2)$$

$$\text{The solution to} \quad \dot{\underline{y}} = J\underline{y} \quad \text{is} \quad \underline{y} = C_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{The solution to} \quad \dot{\underline{x}} = A\underline{x} \quad \text{is} \quad \underline{x} = C_1 e^{\lambda_1 t} \underline{E}_1 + C_2 e^{\lambda_2 t} \underline{E}_2$$

Type II If $\lambda_1 = \lambda_2$ and $\lambda_1 \in \mathbb{R}$:

$$A = PJP^{-1} \qquad \text{with} \qquad J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \qquad \text{and} \qquad P = (\underline{E}_1, \underline{J}_1)$$

with

$$(A - \lambda_1 I)\underline{J}_1 = \underline{E}_1$$

$$\text{The solution to} \quad \dot{\underline{y}} = J\underline{y} \quad \text{is} \quad \underline{y} = (C_1 + C_2 t) e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda_1 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{The solution to} \quad \dot{\underline{x}} = A\underline{x} \quad \text{is} \quad \underline{x} = (C_1 + C_2 t) e^{\lambda_1 t} \underline{E}_1 + C_2 e^{\lambda_1 t} \underline{J}_1$$

Type III If $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ and $\alpha, \beta \in \mathbb{R}$ with, $\beta > 0$:

$$A = PJP^{-1} \qquad \text{with} \qquad J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \qquad \text{and} \qquad P = (\underline{e}_1, \underline{e}_2)$$

with

$$\underline{e}_1 = \mathrm{Re}(\underline{E}_1) \qquad \text{and} \qquad \underline{e}_2 = \mathrm{Im}(\underline{E}_1)$$

$$\text{The solution to} \quad \dot{\underline{y}} = J\underline{y} \quad \text{is} \quad \underline{y} = r_0 e^{\alpha t} \cos(-\beta t + \theta_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0 e^{\alpha t} \sin(-\beta t + \theta_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{The solution to} \quad \dot{\underline{x}} = A\underline{x} \quad \text{is} \quad \underline{x} = r_0 e^{\alpha t} \cos(-\beta t + \theta_0)\underline{e}_1 + r_0 e^{\alpha t} \sin(-\beta t + \theta_0)\underline{e}_2$$

Type IV If $A = \lambda I$, proportional to the identity, then

$$\text{The solution to} \quad \dot{\underline{x}} = A\underline{x} \quad \text{is} \quad \underline{x} = C_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$