Solutions to Sheet 4: nonlinear two dimensional systems

1. (a) The Jacobian matrix is defined as

$$A = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{pmatrix}.$$

For system (a) this gives

$$A_a = \begin{pmatrix} 2x_1x_2 + x_2 & x_1^2 + x_1 \\ \cos x_1 \cos x_2 & -\sin x_1 \sin x_2 \end{pmatrix}.$$

For system (b)

$$A_b = \begin{pmatrix} 2x_1x_2^2 + \sin x_2 - e^{x_1} & 2x_1^2x_2 + x_1 \cos x_2 \\ \sin x_2 - x_2 \sin x_1 & x_1 \cos x_2 + \cos x_1 \end{pmatrix},$$

and for system (c)

$$A_c = \begin{pmatrix} 1 + x_2 & 1 + x_1 \\ \sin x_2 - x_2 \sin x_1 & x_1 \cos x_2 + \cos x_1 \end{pmatrix}.$$

(b) The fixed point is the solution to the simultaneous equations

$$X_1(x_1, x_2) = X_2(x_1, x_2) = 0.$$

It is easy to check that for all three systems

$$X_1(0,0) = X_2(0,0) = 0,$$

so the origin is a fixed point.

(c) At the origin, the Jacobian matrices above become

$$A_a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$A_b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A_c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The linearised equations are simply $\dot{\underline{x}} = A\underline{x}$, where $A = A_a$ or $A = A_b$ or $A = A_c$, depending on the equations.

(d) A fixed point is simple, if the determinant of the Jacobian matrix is not zero. From the previous section, it is clear that $\det(A_a) = 0$ and $\det(A_b) = -1$, $\det(A_c) = 1$. Therefore the fixed point at the origin is simple for systems (b) and (c) and nonsimple for system (a).

(e) For system (b), the Jacobian matrix A_b is already in Jordan form. The eigenvalues of the matrix are real, with $\lambda_1 = -1$ and $\lambda_2 = 1$, therefore the fixed point at the origin is a saddle. For system (c), we have to find the eigenvalues of the matrix A_c which are given by the solution to the equation $(1 - \lambda)^2 = 0$. Therefore, it has two equal, positive eigenvalues $\lambda_1 = \lambda_2 = 1$ which means the fixed point is an unstable improper node.

2. (a) The fixed points are the solutions of

$$x_1 \cos x_2 = 0, \qquad x_2 \cos x_1 = 0.$$

The first equation is satisfied for $x_1 = 0$ or $x_2 = \pm\frac{\pi}{2}$.
$x_1 = 0$ solves the second equation also if an only if $x_2 = 0$. Therefore we have a fixed point at $(0, 0)$.
$x_2 = \pm\frac{\pi}{2}$ only solves the second equation if $\cos x_1 = 0$, which gives $x_1 = \pm\frac{\pi}{2}$ as well. Altogether we have five fixed points at

$$(0,0), \quad \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \left(\frac{\pi}{2}, -\frac{\pi}{2}\right), \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right).$$

(b) The Jacobian is given by the matrix

$$A_{(x_1,x_2)} = \begin{pmatrix} \cos x_2 & -x_1 \sin x_2 \\ -x_2 \sin x_1 & \cos x_1 \end{pmatrix}.$$

(c) The linearised systems for each of the fixed points would be of the form,

$$\underline{\dot{x}} = A_{(x_1,x_2)}\underline{x}, \qquad \text{with} \quad (x_1, x_2) \quad \text{a fixed point,}$$

and

$$A_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{(\pm\frac{\pi}{2},\pm\frac{\pi}{2})} = \begin{pmatrix} 0 & -\frac{\pi}{2} \\ -\frac{\pi}{2} & 0 \end{pmatrix}, \quad A_{(\pm\frac{\pi}{2},\mp\frac{\pi}{2})} = \begin{pmatrix} 0 & \frac{\pi}{2} \\ \frac{\pi}{2} & 0 \end{pmatrix}.$$

(d) In order to classify the fixed points we have to study the three matrices obtained in the previous section. The matrix $A_{(0,0)}$ is a particularly easy case because it is the identity matrix. Therefore, the fixed point at the origin must be an unstable star node.
For the other two matrices we need to work out the eigenvalues and eigenvectors. For both matrices we have that the eigenvalues satisfy $\lambda^2 - \frac{\pi^2}{4} = 0$ which gives eigenvalues $\lambda_1 = \frac{\pi}{2}$ and $\lambda_2 = -\frac{\pi}{2}$ in both cases. This means that all other four fixed points in this case are saddles.
It will be useful for the sketch of the phase diagram to find the eigenvectors of these matrices. The eigenvectors of $A_{(\pm\frac{\pi}{2},\pm\frac{\pi}{2})}$ can be computed to

$$\underline{E}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \underline{E}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

whereas for $A_{(\pm\frac{\pi}{2},\mp\frac{\pi}{2})}$ the eigevectors are exactly the same but with the names exchanged.

(e) The equation for $\frac{dx_2}{dx_1}$ is obtained just by dividing the original equations by each other

$$\frac{dx_2}{dx_1} = \frac{x_2 \cos x_1}{x_1 \cos x_2}.$$

This equation is hard to solve in general, but it is possible to find particular solutions like the ones suggested here. For example we can check that if $x_2 = \pm x_1$ then the equation is satisfied because in that case

$$\frac{dx_2}{dx_1} = \pm 1.$$

(f) When $x_2 = 0$ (the x_1-axis) the equation for $\dot{x}_1$ becomes just $\dot{x}_1 = x_1$ which is solved by $x_1 = C_1 e^t$, where $C_1 > 0$ for $x_1 > 0$ and $C_1 < 0$ for $x_1 < 0$.

(g) The phase diagram is shown in figure 1.

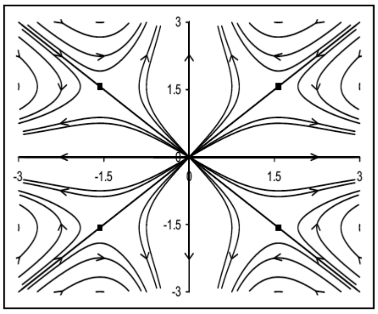

Figure 1: The phase diagram shows the unstable star node at the origin, with the four saddles around it. We can see how the trajectories nicely merge in the regions between fixed points.

3. (a) The new equations are

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_2(1 - 3x_1^2 - 2x_2^2) - x_1,$$

the first of which is linear, whereas the second equation is clearly nonlinear.

(b) In order to linearize the equations we need to compute the Jacobian matrix at a generic point first

$$A_{(x_1,x_2)} = \begin{pmatrix} 0 & 1 \\ -6x_1x_2 - 1 & 1 - 3x_1^2 - 6x_2^2 \end{pmatrix}.$$

At the origin this is the matrix

$$A_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

So, the linearised equations are $\dot{\underline{x}} = A_{(0,0)}\underline{x}$.

(c) To classify the fixed point we need to find the eigenvalues of the Jacobian matrix:

$$-\lambda(1-\lambda) + 1 = 0, \quad \Leftrightarrow \quad \lambda_1 = \frac{1 + i\sqrt{3}}{2}, \lambda_2 = \frac{1 - i\sqrt{3}}{2}.$$

This means that the fixed point is an unstable focus. A pair of eigenvectors associated to these eigenvalues are

$$\underline{E}_1 = \begin{pmatrix} \frac{1-i\sqrt{3}}{2} \\ 1 \end{pmatrix}, \qquad \underline{E}_2 = \begin{pmatrix} \frac{1+i\sqrt{3}}{2} \\ 1 \end{pmatrix}$$

The matrix P that relates $A_{(0,0)}$ to its Jordan form can be written in terms of the real and imaginary parts of $\underline{E}_1$ as

$$P = (\underline{e}_1, \underline{e}_2) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & 0 \end{pmatrix},$$

where $\underline{e}_1 = \mathrm{Re}(\underline{E}_1)$ and $\underline{e}_2 = \mathrm{Im}(\underline{E}_1)$. The Jordan form is

$$J = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

The solution to the canonical equations $\dot{\underline{y}} = J\underline{y}$ is

$$\underline{y} = r_0 e^{\frac{t}{2}} \cos(-\frac{\sqrt{3}}{2}t + \theta_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0 e^{\frac{t}{2}} \sin(-\frac{\sqrt{3}}{2}t + \theta_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where r_0, θ_0 are arbitrary constants. The solution to the linearised equations $\dot{\underline{x}} = A_{(0,0)}\underline{x}$ is

$$\underline{x} = P\underline{y} = r_0 e^{\frac{t}{2}} \cos(-\frac{\sqrt{3}}{2}t + \theta_0)\underline{e}_1 + r_0 e^{\frac{t}{2}} \sin(-\frac{\sqrt{3}}{2}t + \theta_0)\underline{e}_2,$$

or, in components

$$x_1 = \frac{1}{2} r_0 e^{\frac{t}{2}} \cos(-\frac{\sqrt{3}}{2}t + \theta_0) - \frac{\sqrt{3}}{2} r_0 e^{\frac{t}{2}} \sin(-\frac{\sqrt{3}}{2}t + \theta_0), \quad x_2 = r_0 e^{\frac{t}{2}} \cos(-\frac{\sqrt{3}}{2}t + \theta_0).$$

The phase diagram associated to the linearised system would look as in figure 2.

(d) For an unstable focus we find that trajectories periodically cross the axes. The vector field diagram shows however, that far enough from the origin a different behaviour will emerge (see figure 3).

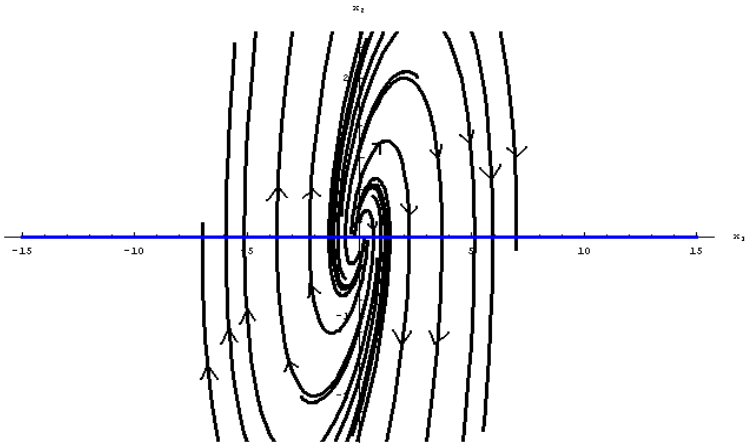

Figure 2: The phase diagram shows the unstable focus at the origin. The blue line (the x_1-axis) corresponds to the line at which all trajectories becomes vertical. This is because $\dot{x}_1 = 0$ in this case corresponds to $x_2 = 0$. Because the focus is unstable all arrows point away from the origin. Trajectories move clockwise as $\dot{x}_1 > 0$ whenever $x_2 > 0$.

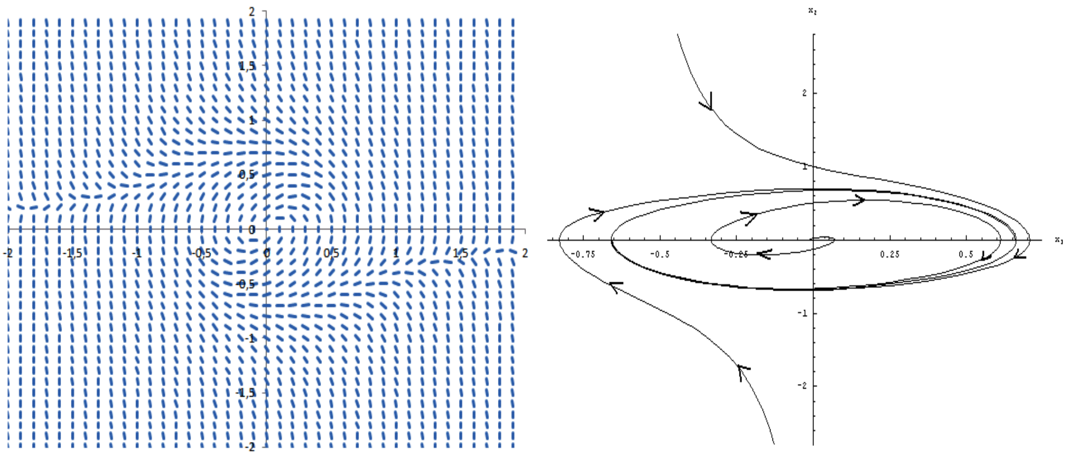

Figure 3: The vector field diagram shows the behaviour of a focus close enough to the origin. However, for trajectories a bit further from the origin we see that instead of crossing the axes they move towards $|x_2| \to \infty$. A picture of some of these trajectories can be seen in the r.h.s. figure. Some "exact" phase space trajectories are shown in that picture. You can work out the directions of the arrows by using the equation $\dot{x}_1 = x_2$. The equation tells you that for $x_2 > 0$, the arrows must point in the direction of increasing x_1 whereas for $x_2 < 0$ the arrow must point in the direction of decreasing x_1. For $t \to \infty$ the trajectories tend to stay in some closed trajectory that spins around the origin.