1. (a) [10 points] Sketch the region bounded by the curves $y = -1$, $y = 1$, $y - x = 3$ and $x = y^2$. Let this region be the integration region R for the integral below. Evaluate the integral.

$$I = \int\int_R (x + y) dx dy.$$

(b) [10 points] Use spherical coordinates to compute the volume of the region bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the cones $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{\frac{x^2+y^2}{3}}$.

Hint: Recall that the spherical coordinates are $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\phi$ and that the element of volume in these coordinates is given by $dxdydz = r^2 \sin\phi dr d\phi d\theta$.

2. (a) [10 points] Find and classify the stationary points (maxima, minima and saddle points) of the function

$$f(x, y) = y \sin x,$$

in the region $0 \leq x \leq 2\pi$.

(b) [10 points] Given the function

$$g(x, y) = \frac{1}{x^2 + y^2 - 1},$$

Write down an expression for the first order differential dg. Use the differential to estimate the value of the function at the point $(1.01, -1.003)$.

Turn over...

3. (a) [8 points] Consider the problem of finding the maximum value of a given function $f(x, y)$ subject to a constraint of the form $\phi(x, y) = 0$. Explain the method of Lagrange multipliers and how it can be applied to solve this kind of problem.

   (b) [12 points] Use the method of Lagrange multipliers to solve the problem of part (a) if $f(x, y) = x^3 + y^3$ and $\phi(x, y) = x + y - 9$. Compute also the value of the Lagrange multiplier.

4. (a) [7 points] Find the general solution $y(x) = c_1 u_1(x) + c_2 u_2(x)$ of the following homogeneous second-order differential equation

$$y'' - y = 0,$$

   where c_1, c_2 are arbitrary constants. Compute the Wronskian of u_1, u_2.

   (b) [13 points] Use the method of variation of parameters to find a particular solution of the inhomogeneous second-order differential equation

$$y'' - y = \frac{1}{1 + e^{-x}}.$$

   Hence determine the general solution of this inhomogeneous equation.

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