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CITY UNIVERSITY

London

BSc Honours Degrees in Mathematical Science
BSc Honours Degree in Mathematical Science with Finance and Economics
BSc Honours Degree in Actuarial Science
BSc Honours Degree in Statistical Science with Management Studies

Part II Examination

Calculus and Linear Algebra

Monday 11 June 2001 1:00 pm – 4:00 pm

Time allowed: 3 hours

Full marks may be obtained for correct answers to FIVE of the EIGHT questions with not more than THREE questions from either section

If more than FIVE questions are answered, the best FIVE marks will be credited.

Use a separate answer book for each section.

Section A: Calculus

1. (a) Sketch the region of integration in the double integral

$$I = \int_0^1 dx \int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right) dy.$$

By changing the order of integration, evaluate I.

(b) Show that in polar coordinates the equation of the circle $(x-1)^2 + y^2 = 1$ takes the form $r = 2\cos\theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
Hence, by using the cylindrical coordinates, find the volume of the solid enclosed in the vertical cylinder $(x-1)^2+y^2 = 1$ bounded below by the plane $z = 0$ and bounded above by the cone $z = 2-\sqrt{x^2+y^2}$.

2. (a) Show that the function

$$f(x,y) = -xy\, e^{-\frac{x^2+y^2}{2}}$$

has a stationary point at $x = 0$, $y = 0$.
Find all the other stationary points of $f(x,y)$.
Determine the nature of the stationary point at $(0,0)$.

(b) Use Taylor's theorem to expand the function $f(x,y) = x^3y^3$ up to second order in the displacements h, k from the point $(1,1)$.
Verify your result directly by setting $x = 1+h$ and $y = 1+k$ in the function $f(x,y)$.

Turn over ...

3. Determine functions $y_1(x)$ and $y_2(x)$ in order that $y(x) = Ay_1(x)+By_2(x)$ is the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0,$$

where A, B are arbitrary constants. Show that the Wronskian of the functions $y_1(x)$ and $y_2(x)$ is nowhere zero.

Use the method of variation of constants to find a particular solution of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \frac{e^{-x}}{\cos^3 x}.$$

Hence determine the general solution of this inhomogeneous equation.

4. (a) Using the method of Lagrange's multipliers, find the shortest distance from the point $P(0, 0, 1)$ to the curve $y^3 + x^3 + y^2 + x^2 = 0$ which lies in the (x, y)-plane.

   (b) Given that $F(x, y, z) = 0$ defines z implicitly as a function of x and y, derive the formulae for $\partial z/\partial x$ and $\partial z/\partial y$ in terms of partial derivatives of F.

   If $z - xy^2z^3 - 2xyz = 0$, determine $\partial z/\partial x$ and $\partial z/\partial y$.

   Given that $z - ax + by + c = 0$ is the plane tangent to the surface $z - xy^2z^3 - 2xyz = 0$ at the point $(-1, -1, 1)$, find the values of a, b and c.

Section B: Linear Algebra

5. (a) Clearly explaining your notation, show how any system of m real linear equations in n unknowns may be encoded as a matrix equation.

   (b) The symbol $\mathbb{R}$ denotes the set of real numbers. Explain what is meant by the set denoted $\mathbb{R}^n$.

   (c) Explain how $\mathbb{R}^n$ may be equipped with the property of vector space over $\mathbb{R}$.

   (d) Show that the solution set to your system in (a) above is a subspace of $\mathbb{R}^n$ if and only if the system is homogeneous.

   (e) For V a vector space over $\mathbb{R}$, give the definition of *inner product* on V. Give an example of an inner product on $\mathbb{R}^n$.

6. The symbol $\mathbb{R}$ denotes the set of real numbers. Let V denote the vector space over $\mathbb{R}$ of 4–component real row vectors.

   (a) Determine which of the following is a basis for V, giving reasons for each answer.

   (i)
$$S_1 = \{(1, 2, 3, 4),\ (5, 6, 7, 8),\ (9, 120, 11, 2)\}$$

   (ii)
$$S_2 = \{(1, 0, 0, 0),\ (1, 1, 0, 0),\ (1, 1, 1, 0),\ (1, 1, 1, 1)\}$$

   (iii)
$$S_3 = S_1 \cup S_2$$

   (iv)
$$S_4 = \{(1, 2, 3),\ (0, 1, 0),\ (0, 0, 1)\}$$

   (b) For each set S_i in (a) above which is a linearly independent subset of V determine a set S such that $S_i \cup S$ is a basis.

   (c) Give a basis for the row space of the matrix
$$M = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}.$$

   Give a basis for the column space of M, and hence confirm that the row space and the column space have the same dimension. Construct an explicit linear map between these two vector spaces which is an isomorphism.

Turn over ...

7. (a) The weather in Toontown is always either snowy or sunny. A statistical analysis shows that if it is snowy on any given day, then there is a 50% chance it will be snowy again on the following day; but if it is sunny on one day then there is only a 1/3 chance it will be snowy the next. Given that it is sunny on day 1 of a certain sequence of days, let

$$v^i = \begin{pmatrix} v^i_{snowy} \\ v^i_{sunny} \end{pmatrix}$$

be such that v^i_x is the probability of weather condition x on day i (so $v^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$). Give a matrix equation relating v^{i+1} to v^i, and show that there is a theoretical "steady state" value of v^i, i.e., such that $v^i = v^{i+1}$. Determine this value (by solving a 2×2 matrix eigenvalue equation or otherwise), and explain why there *is* only one such value.

(b) Determine the eigenvalues and three independent eigenvectors of the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Turn over ...

8. (a) We are told that a quadratic function

$$g(x) = g_0 + g_1 x + g_2 x^2 \tag{1}$$

obeys constraints $g(1) = 1$, $g(2) = 2$ and $g(3) = 1$, but we are not told the value of the coefficients, g_0, g_1, g_2, of the quadratic. Inserting the constraint data into equation (1) we learn immediately that

$$g_0 + g_1 + g_2 = 1,$$

$$g_0 + 2g_1 + 4g_2 = 2,$$

$$g_0 + 3g_1 + 9g_2 = 1.$$

Use these facts to show that there is exactly one quadratic function obeying the constraints, i.e., that they completely determine the coefficients g_0, g_1, g_2, and compute this quadratic.

(b) Let P_j denote the price of a certain commodity on the j^{th} day of commodity trading in some market. You are given the prices $P_1, P_2, ..., P_n$, i.e., the prices on days 1 to n. Show in the case $n = 4$ that there exists a polynomial

$$f(x) = \sum_{i=0}^{n-1} f_i x^i$$

such that $f(j) = P_j$ for each given P_j. Note that it is NOT necessary to determine $f(x)$.

Having performed such an analysis on data for each of two separate commodities an analyst finds that the polynomial for commodity A, let us call it f_A, is of degree $n-1$, while there is a polynomial f_B of degree $m < n-1$ fitting the data for commodity B. Which of these polynomials would you expect to be likely to give a better prediction of the corresponding commodity's price on day $j+1$? Give reasons for your answer.

Internal Examiners:	Professor J. Mathon
	Professor P.P. Martin
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