528.50

CITY UNIVERSITY

London

BSc Honours Degrees in Mathematical Science
BSc Honours Degree in Mathematical Science with Finance and Economics
BSc Honours Degree in Actuarial Science
BSc Honours Degree in Statistical Science with Management Studies

PART II EXAMINATION

Calculus and Linear Algebra

Tuesday 28 May 2002 1:00 pm – 4:00 pm

Time allowed: 3 hours

Full marks may be obtained for correct answers to FIVE of the EIGHT questions with not more than THREE questions from either section

If more than FIVE questions are answered, the best FIVE marks will be credited.

Use a separate answer book for each section.

Section A: Calculus

1. Determine functions $y_1(x)$ and $y_2(x)$ in order that $y(x) = Ay_1(x) + By_2(x)$ is the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0,$$

where A, B are arbitrary constants. Show that the Wronskian of the functions $y_1(x)$ and $y_2(x)$ is nowhere zero.

Use the method of variation of constants to find a particular solution of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = \frac{2e^{2x}}{\sin x}.$$

Hence determine the general solution of this inhomogeneous equation.

2. Assuming that the solution of the differential equation

$$\frac{d^2y}{dx^2} + x^2y = 0$$

can be expressed as a power series

$$\sum_0^\infty a_n x^n,$$

show that the coefficients a_n satisfy the recurrence relation

$$a_{n+2} = \frac{-a_{n-2}}{(n+1)(n+2)}; \quad n = 2, 3, ...$$

Hence determine the first four non-zero terms of a general series solution of the differential equation.

Turn over . . .

3. (a) Sketch the region of integration in the double integral

$$I = \int_0^1 dy \int_y^1 \cos\left(\frac{1}{2}\pi x^2\right)\, dx.$$

By changing the order of integration, evaluate I.

(b) Find the Jacobian of the transformation $x = v/u$ and $y = v$. Using the variables u, v evaluate the double integral

$$\int\!\!\int_S \frac{y^2}{x^2}\, e^{y/x}\, dx\, dy,$$

where S is the region defined by $0 \leq x \leq 1$, $0 \leq y \leq x$.

4. (a) Use Taylor's theorem to expand the function

$$f(x, y) = e^{2x+3y}(8x^2 - 6xy + 3y^2)$$

up to second-order terms in the components h, k of the displacements around the origin $(0, 0)$. What can you conclude from the form of the expansion about the nature of the point $(0, 0)$?

(b) Using the method of Lagrange's multipliers, find the shortest distance from the point $P(0, 0, 1)$ in three dimensional space to the curve $y^2 + x^2 + 4xy - 4 = 0$ which lies in the (x, y)-plane $(z = 0)$.

Turn over ...

Section B: Linear Algebra

5. (a) Determine which of the following maps are linear (giving reasons for your answers).

   (i) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad (x_1, x_2, x_3) \longmapsto (x_1 + x_2, x_1 - x_3, x_2 x_3)$.

   (ii) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad (x_1, x_2, x_3) \longmapsto (x_1 - x_3, x_2 + x_1 + 2x_3, 3x_1)$.

   (iii) $f : \mathbb{R}^3 \longrightarrow \mathbb{R} \qquad (x_1, x_2, x_3) \longmapsto 1 + 2x_1 + 3x_2 + 4x_3$.

   (b) Define the rank and nullity of a linear map, and state carefully a theorem that relates the two.

   (c) Let $M(2,2)$ denote the vector space over $\mathbb{R}$ of all real-valued 2×2 matrices, and $f : M(2,2) \longrightarrow M(2,2)$ be the map

$$f : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a+c & 0 \\ b+c+d & c+d \end{pmatrix}.$$

   Find bases for the image and kernel of f, stating carefully any theorems that you use. (You may assume that f is linear, and may use without proof that $\dim M(2,2) = 4$.)

6. Let A be the matrix

$$\begin{pmatrix} 5 & -15 & 3 \\ 3 & -13 & 3 \\ 9 & -45 & 11 \end{pmatrix}.$$

   By finding a basis of eigenvectors, determine invertible matrices P and P^{-1} such that $P^{-1}AP$ is diagonal. Hence write down three matrices whose product equals A^n.

7. Let $P_2(x)$ denote the real vector space of all polynomials of degree at most 2 in x with real coefficients.

   (a) Show that $\langle p, q \rangle = \int_0^1 p(x)q(x)dx$ defines an inner product on $P_2(x)$.

   (b) Use the Gram-Schmidt method to form an orthonormal basis for $P_2(x)$ (with respect to the above inner product) from the set of basis elements $\{1, x, x^2\}$.

   (c) Given an arbitrary polynomial $p \in P_2(x)$, write p as a linear combination of your new basis elements. Verify your formula in the case $p = x + 1$.

Turn over ...

8. In this question you may assume any standard dimension results from the course that you require.

   (a) State what it means for a set of vectors in a vector space V over $\mathbb{F}$ to be

      (i) linearly independent;

      (ii) spanning.

   For a finite-dimensional vector space V state a relation that must hold between the size of any spanning set and the size of any linearly independent set.

   (b) Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a basis for some vector space V over $\mathbb{R}$. For each of the following sets, determine whether it is linearly independent, spanning, both, or neither. Give reasons for your answers.

      (i) $\{\mathbf{u} + 2\mathbf{v}, \mathbf{v} - \mathbf{w}\}$.

      (ii) $\{\mathbf{u} + \mathbf{w}, \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} - \mathbf{w}\}$.

   (c) (i) State (without proof) the dimension of $P_n(x)$, the real vector space of all polynomials of degree at most n in x with real coefficients.

      (ii) Let V be the vector space over $\mathbb{R}$ of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$. Show that V is not finite dimensional.

Internal Examiners:	Professor J. Mathon
	Dr A.G. Cox
External Examiners:	Professor S.A. Robertson
	Professor D.J. Needham