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CITY UNIVERSITY

London

BSc Honours Degrees in Mathematical Science
BSc Honours Degree in Mathematical Science with Finance and Economics
BSc Honours Degree in Actuarial Science
BSc Honours Degree in Statistical Science with Management Studies

PART II EXAMINATION

Calculus and Linear Algebra

Monday 7 June 1999 1:00 pm – 4:00 pm

Time allowed: 3 hours

Full marks may be obtained for correct answers to FIVE of the EIGHT questions with not more than THREE questions from either section

If more than FIVE questions are answered, the best FIVE marks will be credited.

Use a separate answer book for each section.

Section A

1. (a) Sketch the region of integration in the double integral

$$I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)} dy.$$

By transforming to polar coordinates, evaluate I.

(b) The region R in the positive octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) is bounded by the surface $y = 4x^2$ and by the planes $x = 0$, $y = 4$, $z = 0$ and $z = 2$. Evaluate the volume integral

$$\int\int\int_R 2x \, dxdydz.$$

2. (a) Find and classify the stationary points of the function

$$f(x, y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x.$$

(b) Use Taylor's theorem to expand the function $f(x, y) = (x+y)e^{(x-y)}$ up to second-order terms in the components h, k of the displacements around the point $(-1, -1)$. Hence estimate the value of the function f at the point $(-0.9, -1.05)$.

Turn over ...

3. Determine functions $y_1(x)$ and $y_2(x)$ in order that $y(x) = Ay_1(x) + By_2(x)$ is the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} + y = 0,$$

where A, B are arbitrary constants. Show that the Wronskian of the functions $y_1(x)$ and $y_2(x)$ is nowhere zero.

Use the method of variation of constants to find a particular solution of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + y = x + \frac{1}{cos(x)}.$$

Hence determine the general solution of this inhomogeneous equation.

4. (a) A change of variables $(u, v) \longmapsto (x, y)$ is defined by

$$x = \frac{1}{2}(u + v), \ y = \frac{1}{4}(u^2 + v^2).$$

If $f(x, y)$ is a twice differentiable function and $f(x(u, v), y(u, v)) = F(u, v)$, show that

$$\frac{\partial F}{\partial u} + \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

Show also that

$$\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} + x\frac{\partial^2 f}{\partial x \partial y} + y\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y}.$$

(b) Given that $F(x, y, z) = 0$ defines z implicitly as a function of x and y, derive formulae for $\partial z/\partial x$ and $\partial z/\partial y$ in terms of partial derivatives of F.
If $cos(x + y) + sin(y + z) = 1$, determine $\partial z/\partial x$ and $\partial z/\partial y$.

Turn over ...

Section B

5. (a) Show that the following mappings are linear, and determine the inverse map, where appropriate:

   (i)
$$M : \mathbb{R}^2 \to \mathbb{R}^2$$
   defined by
$$M : (x, y) \mapsto (3x + y, x)$$
   (ii)
$$N : \mathbb{R}^3 \to \mathbb{R}$$
   defined by
$$N : (x, y, z) \mapsto 3x + y + z$$

   (b) Show that the following mappings are not linear:

   (i)
$$S : \mathbb{R}^2 \to \mathbb{R}^2$$
   defined by
$$S : (x, y) \mapsto (3x + y, x + 1)$$
   (ii)
$$T : \mathbb{R}^3 \to \mathbb{R}$$
   defined by
$$T : (x, y, z) \mapsto 3x + yz$$

   (c) Let $M : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by
$$M(x, y, z, t) = (x + 2y - z, y + z, x + y - 2z, t).$$
   Find bases for the image and kernel of M (you may assume M to be linear).

Turn over...

6. Let n be a fixed but unspecified positive integer, and let $\mathcal{M}$ be the set of all $n \times n$ matrices with complex coefficients.

   (a) Explain what is meant by a similarity transformation on an element of $\mathcal{M}$. Show that similarity is an equivalence relation on $\mathcal{M}$.

   (b) Explain what is meant by the Jordan canonical form of a matrix in $\mathcal{M}$.

   (c) State the Cayley-Hamilton theorem, and prove it for the case of an $n \times n$ matrix with n distinct eigenvalues.

   (d) Determine the eigenvalues and four corresponding linearly independent eigenvectors of

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

7. Let D_n denote the set of real $n \times n$ matrices for which the vector $(1, 1, \cdots, 1)^t$ is an eigenvector, with eigenvalue 0. Show that D_n is a subspace of the space $M_n(\mathbb{R})$ of all real $n \times n$ matrices over $\mathbb{R}$. An equivalence relation on $M_n(\mathbb{R})$ is defined by $A \rho B$ if $A - B \in D_n$. Explain how operations of addition of equivalence classes of ρ and multiplication of equivalence classes of ρ by a scalar may be defined, and prove that with these operations the set of all equivalence classes of ρ becomes a vector space over $\mathbb{R}$.

8. Let V be the vector space (over $\mathbb{R}$) of all real 2×2 matrices and let

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

   (a) Verify that the set $\{E_1, E_2, E_3, E_4\}$ is a basis for V.

   (b) An inner product on V is given by $< u, v > = \text{trace}(v^t u)$. Determine an orthonormal basis of V with respect to $<,>$.