2.5.2 Standard coordinate systems in $\mathbb{R}^2$ and $\mathbb{R}^3$

Similarly as for functions of one variable, integrals of functions of two or three variables may become simpler when changing coordinates in an appropriate way. For this reason we are going to introduce in this section the so-called standard coordinate systems which are used in 2 and 3-dimensions. Knowing these coordinate systems will allow us in the next section to perform changes of variables in volume integrals.

Coordinate systems in $\mathbb{R}^2$
There are two standard coordinate systems which are used to describe points in 2-dimensional space. These coordinate systems are

- the **Cartesian coordinate system** (which we normally use), in which we characterize points by two coordinates (x, y) and
- the **Polar coordinate system** in which we characterize points in the 2-dimensional xy-plane by their distance to the origin r and the angle θ (see figure 21).

As you see in the picture, both coordinate systems are related by the transformation

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{2.303}$$

with $0 \leq \theta \leq 2\pi$, or equivalently

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{2.304}$$

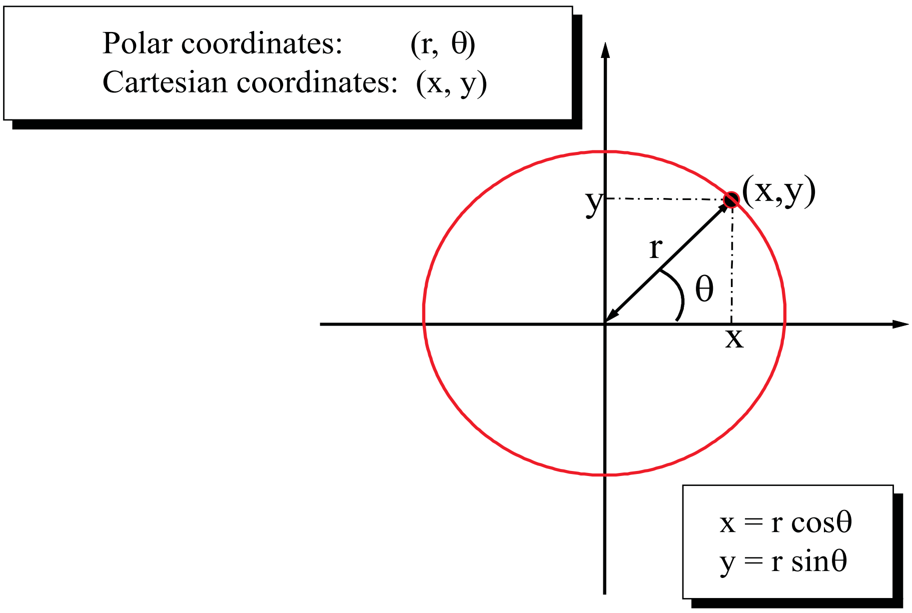

Figure 21: *The polar and Cartesian coordinate systems.*

Coordinate systems in $\mathbb{R}^3$
There are three standard coordinate systems which are used to describe points in 3-dimensional space. These coordinate systems are

- the **Cartesian coordinate system** (which we normally use), in which we characterize points by three coordinates (x, y, z) and
- the **cylindrical coordinate system**: this coordinate system is a sort of generalization of polar coordinates in two dimensions. In cylindrical coordinates a point in 3-dimensional space is characterized by coordinates (r, θ, z), which are defined as shown in figure 22 (they are the same as in polar coordinates plus one extra coordinate describing the height in the z direction),
- the **spherical coordinate system**, in which a point in 3-dimensional space is characterized by the distance to the origin r and the angles θ, ϕ defined in figure 23,

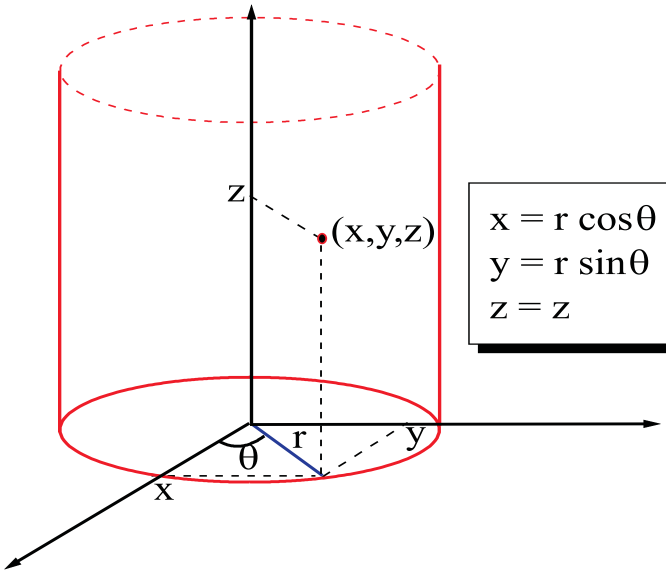

Figure 22: *The cylindrical and Cartesian coordinate systems.*

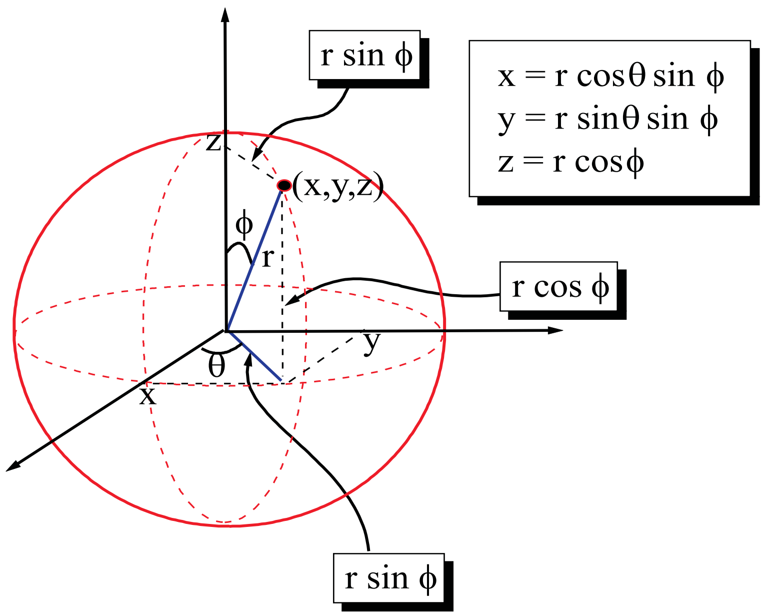

Figure 23: *The spherical and Cartesian coordinate systems.*

The relation between cylindrical and spherical coordinates and Cartesian coordinates is given in the figures 21 and 23. For cylindrical coordinates we have

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z, \tag{2.305}$$

with $0 \leq \theta \leq 2\pi$ or equivalently

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right), \qquad z = z. \tag{2.306}$$

For spherical coordinates we have

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi, \tag{2.307}$$

with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$ or equivalently

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right), \qquad \phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right). \tag{2.308}$$

<u>**Note:**</u> **When is it convenient to use polar, cylindrical or spherical coordinates instead of Cartesian coordinates?** Essentially it depends on the characteristics of the integration region they are asking us to consider. For example, suppose we are asked to solve the following problem: **compute the volume of a sphere of radius 3**, characterized by the equation

$$x^2 + y^2 + z^2 = 9. \tag{2.309}$$

We can try to solve the problem in the same way we have seen in the previous examples. We must sketch the region of integration (see figure 23), then determine the integration region in x, y and z and compute the integral

$$\int\int\int_R dxdydz. \tag{2.310}$$

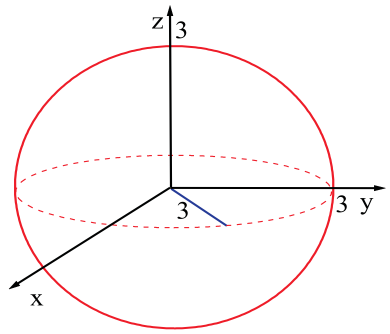

Figure 24: *The integration region of our problem.*

In terms of the coordinates x, y and z, the integration region is relatively complicated. In fact it is given by

$$R = \Big\{(x,y,z) : -3 \leq x \leq 3, \quad -\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2},$$
$$-\sqrt{9-x^2-y^2} \leq z \leq \sqrt{9-x^2-y^2}\Big\}. \tag{2.311}$$

If we now try to compute the integral we will soon see that it becomes rather complicated

$$V = \int_{x=-3}^{x=3} dx \int_{y=-\sqrt{9-x^2}}^{x=\sqrt{9-x^2}} dy \int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}} dz. \tag{2.312}$$

The first integral gives

$$\int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}} dz = [z]_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} = 2\sqrt{9-x^2-y^2}. \tag{2.313}$$

Inserting this result into the integral in y we have

$$\int_{y=-\sqrt{9-x^2}}^{x=\sqrt{9-x^2}} 2\sqrt{9-x^2-y^2}dy. \tag{2.314}$$

This integral is not completely trivial to do. The best way to do it is to change coordinates as

$$y = \sqrt{9-x^2}\cos\alpha \quad \Rightarrow \quad dy = -\sqrt{9-x^2}\sin\alpha d\alpha, \tag{2.315}$$

with $\pi \leq \alpha \leq 0$. Changing coordinates that way, the integral (2.314) becomes

$$\begin{aligned} -\int_{\alpha=\pi}^{\alpha=0} 2(9-x^2)\sin^2(\alpha)d\alpha &= \int_{\alpha=0}^{\alpha=\pi} (9-x^2)(1-\cos(2\alpha))d\alpha \\ &= (9-x^2)\left[\alpha - \frac{\sin(2\alpha)}{2}\right]_0^{\pi} = (9-x^2)\pi. \end{aligned} \tag{2.316}$$

Finally, integrating in x we obtain the volume

$$\begin{aligned} V &= \pi\int_{x=-3}^{x=3}(9-x^2)dx = \pi\left[9x - \frac{x^3}{3}\right]_{-3}^{3} \\ &= \pi\left(27 - \frac{27}{3}\right) - \pi\left(-27 + \frac{27}{3}\right) = 36\pi. \end{aligned} \tag{2.317}$$

Therefore, the final result is 36π which is indeed the volume of a sphere of radius 3[‡]. This way of computing the volume is correct but it is in fact much more complicated than if we had used spherical coordinates from the beginning. In that case, the integration region is very easy to determine

$$R = \{(x,y,z) : 0 \leq r \leq 3, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi\}, \tag{2.318}$$

and **the only difficulty is to determine how $dx\,dy\,dz$ can be expressed in terms of $dr\,d\theta\,d\phi$**. We will see in the next section how this relation can be found. The result we are going to find is

$$dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi. \tag{2.319}$$

[‡]Remember that the volume of a sphere of radius r is given by:

$$V = \frac{4}{3}\pi r^3.$$

If we know this, then we can compute our integral very easily. It is just

$$\begin{aligned} V &= \int_{r=0}^{r=3} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\phi=0}^{\phi=\pi} \sin\phi \, d\phi \\ &= \int_{r=0}^{r=3} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta \, [-\cos\phi]_0^{\pi} = 2 \int_{r=0}^{r=3} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta \\ &= 2 \int_{r=0}^{r=3} r^2 dr \, [\theta]_0^{2\pi} = 4\pi \int_{r=0}^{r=3} r^2 dr = 4\pi \left[\frac{r^3}{3} \right]_0^3 = 4\pi \frac{27}{3} = 36\pi. \end{aligned} \tag{2.320}$$

We can say as a conclusion that whenever the integration region is a sphere (or part of a sphere), we must use spherical coordinates. If the integration region is a cylinder (or part of a cylinder), we must use cylindrical coordinates. If the integration region is a disk (or part of one) it is best to use polar coordinates.