2.5.3 Change of variables and Jacobians

In the previous example we saw that, once we have identified the type of coordinates which is best to use for solving a particular problem, the next step is to do the change of coordinates on the integral we want to compute. One way to see how this goes, is to draw a picture of an infinitesimal element of volume (or surface, if we are doing an integral of a function of two variables) and compute its volume (surface) in terms of the new variables. Let us do that for the simplest case of two variables.

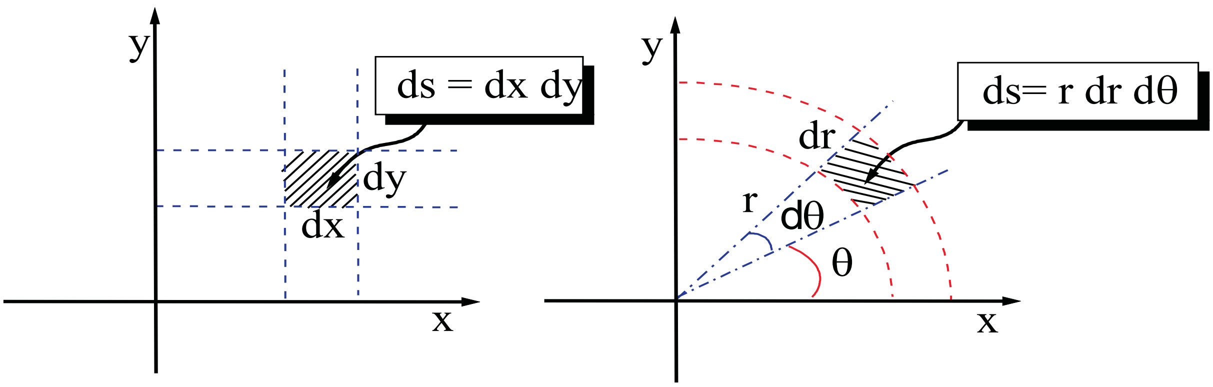

Figure 25: *Differentials of surface.*

Consider an infinitesimal rectangle in Cartesian coordinates. Its area is given by $ds = dx\,dy$. What is the surface of an elementary infinitesimal region in polar coordinates? The answer follows from figure 25, that is

$$dx\,dy = r\,d\theta\,dr. \tag{2.321}$$

Therefore, given an integral

$$I = \int\int_R f(x,y)dxdy, \tag{2.322}$$

a change to polar coordinates will give

$$I = \int\int_{R'} f(r\cos\theta, r\sin\theta)rd\theta dr, \tag{2.323}$$

where R' is the integration region R in terms of the new coordinates.

Example: Compute the integral

$$I = \int\int_R \sqrt{x^2+y^2}dxdy, \tag{2.324}$$

on a disk of radius a. This is a typical case in which the best is to use polar coordinates (the integration region is a disk!). In polar coordinates

$$R = \{(r,\theta) : 0 \leq r \leq a, \qquad 0 \leq \theta \leq 2\pi\}. \tag{2.325}$$

In addition, we have seen before that

$$\sqrt{x^2+y^2} = r, \qquad dx\,dy = r\,dr\,d\theta, \tag{2.326}$$

therefore, the integral in polar coordinates is simply

$$I = \int_{r=0}^{r=a} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta. \tag{2.327}$$

The integral in θ is just

$$\int_{\theta=0}^{\theta=2\pi} d\theta = [\theta]_0^{2\pi} = 2\pi. \tag{2.328}$$

So we finally get

$$I = 2\pi \int_{r=0}^{r=a} r^2 dr = 2\pi \left[\frac{r^3}{3}\right]_0^a = \frac{2\pi a^3}{3}. \tag{2.329}$$

We have found the result (2.321) from geometrical considerations (from figure 25). This can be done for any change of coordinates, in 2 or 3 dimensions. However there is a more systematic way to compute the element of volume or surface under a change of coordinates. In general we have:

Definition: Let (x, y) be the Cartesian coordinates in 2-dimensional space and consider a generic change of variables

$$x = x(u, v), \qquad \text{and} \qquad y = y(u, v), \tag{2.330}$$

(u, v) being the new variables. Then the differentials of surface are related in the following way

$$dx\, dy = |J|\, du\, dv, \tag{2.331}$$

where $|J|$ is the **modulus** of the following determinant

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \tag{2.332}$$

This determinant is called **the Jacobian of the transformation of coordinates**.

Example 1: Use the Jacobian to obtain the relation between the differentials of surface in Cartesian and polar coordinates.

The relation between Cartesian and polar coordinates was given in (2.303). We can easily compute the Jacobian,

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r. \tag{2.333}$$

Therefore

$$dx\, dy = r\, dr\, d\theta, \tag{2.334}$$

which is the same result as (2.321).

Example 2: Find the Jacobian of the transformation

$$x = v/u, \qquad \text{and} \qquad y = v. \tag{2.335}$$

Using these new variables evaluate the integral

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=x} \frac{y^2}{x^2} e^{y/x} dx dy \tag{2.336}$$

We start by computing the Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -v/u^2 & 1/u \\ 0 & 1 \end{vmatrix} = -\frac{v}{u^2}. \tag{2.337}$$

Therefore

$$dx\, dy = |J| du\, dv = \frac{v}{u^2} du\, dv. \tag{2.338}$$

Now we have to transform the function we want to integrate,

$$\frac{y^2}{x^2} e^{y/x} = u^2 e^u, \tag{2.339}$$

and we have to find the new integration region

$$0 \le x \le 1 \quad \Leftrightarrow \quad 0 \le v \le u, \tag{2.340}$$

$$0 \le y \le x \quad \Leftrightarrow \quad 0 \le u \le 1. \tag{2.341}$$

Therefore the integral we need to compute is

$$I = \int_{u=0}^{u=1} e^u du \int_{v=0}^{v=u} v\, dv. \tag{2.342}$$

The first integral is

$$\int_{v=0}^{v=u} v\, dv = \left[\frac{v^2}{2}\right]_{v=0}^{v=u} = \frac{u^2}{2}, \tag{2.343}$$

and so

$$I = \frac{1}{2} \int_{u=0}^{u=1} u^2 e^u du. \tag{2.344}$$

This integral can be done by using integration by parts twice

$$\begin{aligned} \int_{u=0}^{u=1} u^2 e^u du &= \left[u^2 e^u\right]_0^1 - \int_{u=0}^{u=1} 2ue^u du = e - \int_{u=0}^{u=1} 2ue^u du \\ &= e - [2ue^u]_0^1 + \int_{u=0}^{u=1} 2e^u du = e - 2e + \int_{u=0}^{u=1} 2e^u du \\ &= [2e^u]_0^1 - e = 2e - 2 - e = e - 2. \end{aligned} \tag{2.345}$$

Therefore

$$I = \frac{e-2}{2}. \tag{2.346}$$

Definition: Let (x, y, z) be the Cartesian coordinates in 3-dimensional space and consider a generic change of variables

$$x = x(u, v, t), \quad y = y(u, v, t) \quad \text{and} \qquad z = z(u, v, t), \tag{2.347}$$

(u, v, t) being the new variables. Then the differentials of surface are related in the following way

$$dx\, dy\, dz = |J|\, du\, dv\, dt, \tag{2.348}$$

where $|J|$ is the **modulus** of the following determinant

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial t} \end{vmatrix} \tag{2.349}$$

This determinant is called **the Jacobian of the transformation of coordinates**.

Example 1: The Jacobian of cylindrical coordinates.

The relation between Cartesian and cylindrical coordinates was given in (2.305). We can easily compute the Jacobian,

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r. \tag{2.350}$$

Therefore

$$dx\, dy\, dz = |J|\, dr\, d\theta\, dz, = r\, dr\, d\theta\, dz, \tag{2.351}$$

Example 2: The Jacobian of spherical coordinates.

The relation between Cartesian and spherical coordinates was given in (2.307). The Jacobian is,

$$\begin{aligned} J \quad &= \quad \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix} \\ &= \quad -r^2\cos^2\theta\sin^3\phi - r^2\sin^2\theta\cos^2\phi\sin\phi - r^2\cos^2\theta\cos^2\phi\sin\phi \\ &- \quad r^2\sin^2\theta\sin^3\phi = -r^2\sin^3\phi(\cos^2\theta + \sin^2\theta) - r^2\cos^2\phi\sin\phi(\cos^2\theta + \sin^2\theta) \\ &= \quad -r^2\sin^3\phi - r^2\cos^2\phi\sin\phi = -r^2\sin\phi(\sin^2\phi + \cos^2\phi) = -r^2\sin\phi. \end{aligned} \tag{2.352}$$

Therefore

$$dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi = r^2\sin\phi\,dr\,d\theta\,d\phi, \tag{2.353}$$

Let us now see a couple of examples of integral where we use cylindrical and spherical coordinates:

Example 1: Use cylindrical coordinates to evaluate the following integral

$$\int\int\int_R (x^2+y^2)dxdydz, \tag{2.354}$$

where R is the solid bounded by the surface $x^2+y^2=2z$ and the plane $z=2$.

As usual, we start by sketching the integration region. The first equation $x^2+y^2=2z$ is a paraboloid (one of the quadratic surfaces we saw some time ago!). The integration region looks more or less like that

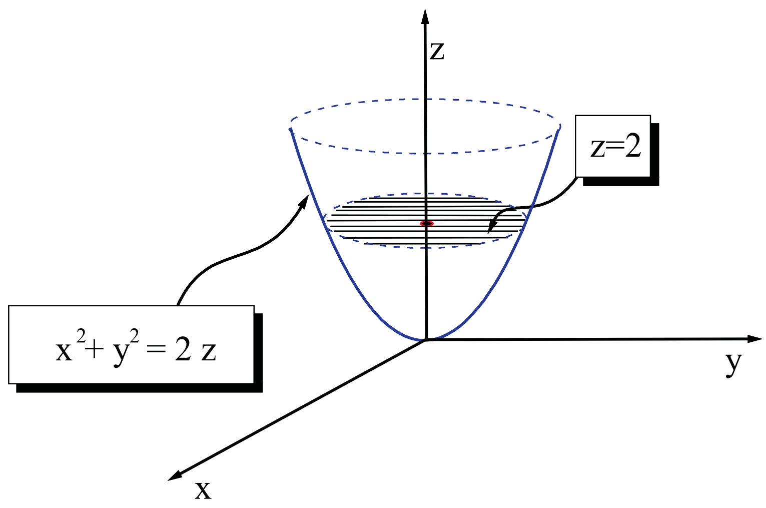

Figure 26: *The integration region of our problem.*

Now we want to do the integral in cylindrical coordinates. We have seen before that

$$x^2+y^2=r^2, \qquad dx\,dy\,dz = r\,dr\,d\theta\,dz \tag{2.355}$$

so the integral we want to compute is

$$\int\int\int_{R'} r^3 dr d\theta dz, \tag{2.356}$$

where R' is the integration region in cylindrical coordinates. From the picture and the information given by the problem it is easy to find

$$R' = \{(r,\theta,z) : 0 \le r \le \sqrt{2z}, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 2\}. \tag{2.357}$$

Therefore our integral is

$$\int_{z=0}^{z=2} dz \int_{\theta=0}^{\theta=2\pi} d\theta \int_{r=0}^{r=\sqrt{2z}} r^3 dr. \tag{2.358}$$

The integral in r is

$$\int_{r=0}^{r=\sqrt{2z}} r^3 dr = \left[\frac{r^4}{4}\right]_0^{\sqrt{2z}} = z^2. \tag{2.359}$$

The integral in θ is simply

$$\int_{\theta=0}^{\theta=2\pi} d\theta = [\theta]_0^{2\pi} = 2\pi. \tag{2.360}$$

Therefore, the final result is

$$2\pi \int_{z=0}^{z=2} z^2 dz = \left[\frac{z^3}{3}\right]_0^2 = \frac{16\pi}{3}. \tag{2.361}$$

Example 2: Use spherical coordinates to compute the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the cone $z = \sqrt{x^2 + y^2}$.

The region of integration for this problem is given in the picture below:

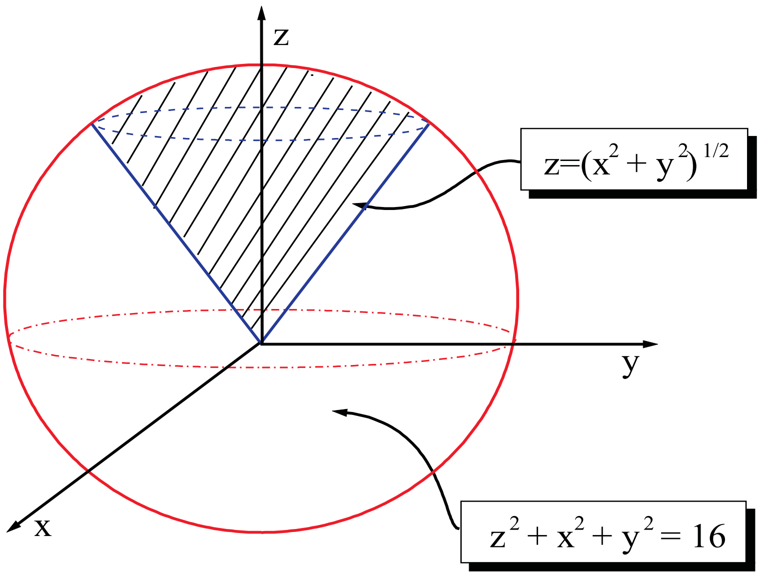

Figure 27: *The sphere $x^2 + y^2 + z^2 = 16$ and the cone $z = \sqrt{x^2 + y^2}$. The dashed region is our integration region.*

The integral we have to compute is

$$\int\int\int_R dxdydz = \int\int\int_{R'} r^2 \sin\phi dr d\theta d\phi. \tag{2.362}$$

Here we have only used the result (2.353) and we called R' the integration in spherical coordinates. In order to determine R' we notice the following: the equation of the sphere in figure 26 in spherical coordinates (see (2.307)) is just

$$x^2 + y^2 + z^2 = r^2 = 16, \tag{2.363}$$

and the equation of the cone is

$$r\cos\phi = \sqrt{r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta} = r\sin\phi, \tag{2.364}$$

from this equation it follows

$$\tan\phi = 1 \quad \Rightarrow \quad \phi = \frac{\pi}{4}. \tag{2.365}$$

This is the angle of the cone with respect to the z axes. Therefore, we just have to integrate in the following region

$$R' = \{(r,\theta,\phi) : 0 \le r \le 4, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/4\}. \tag{2.366}$$

The volume is thus

$$V = \int_{r=0}^{r=4} r^2 dr \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\phi=0}^{\phi=\pi/4} \sin\phi\, d\phi, \tag{2.367}$$

with

$$\int_{\phi=0}^{\phi=\pi/4} \sin\phi\, d\phi = [-\cos\phi]_{\phi=0}^{\phi=\pi/4} = -\frac{1}{\sqrt{2}} + 1 = \frac{2-\sqrt{2}}{2}. \tag{2.368}$$

$$\int_{\theta=0}^{\theta=2\pi} d\theta = [\theta]_0^{2\pi} = 2\pi, \tag{2.369}$$

and

$$\int_{r=0}^{r=4} r^2 dr = \left[\frac{r^3}{3}\right]_0^4 = \frac{64}{3}. \tag{2.370}$$

Therefore the volume is

$$V = \frac{64\pi(2-\sqrt{2})}{3}. \tag{2.371}$$