CALCULUS 1998: EXAM SOLUTIONS

1. (a) The integration region is the triangle formed by the intersection of the lines $y = x$, $y = 0$ and $x = 1$. Once we have identified the integration region, it is easy to change the order of integration to write I equivalently as

$$I = \int_{x=0}^{x=1} dx \int_{y=0}^{y=x} \cos\left(\frac{\pi x^2}{2}\right) dy.$$

The integral

$$\int_{y=0}^{y=x} \cos\left(\frac{\pi x^2}{2}\right) dy = \left[y \cos\left(\frac{\pi x^2}{2}\right)\right]_0^x = x\cos\left(\frac{\pi x^2}{2}\right) - 0 = x\cos\left(\frac{\pi x^2}{2}\right),$$

is trivial to do, since the argument does not depend on y. Now the second integral is also very easy to do, since we have the product of the cosine of a function and the derivative of that function, therefore

$$I = \int_{x=0}^{x=1} x\cos\left(\frac{\pi x^2}{2}\right) dx = \left[\frac{1}{\pi}\sin\left(\frac{\pi x^2}{2}\right)\right]_0^1 = \frac{1}{\pi} - 0 = \frac{1}{\pi}.$$

If you do not realize how to do the integral directly, you can also change variables to $t = \pi x^2/2$ which gives $dt = \pi x dx$ and allows you to rewrite the integral above as

$$I = \int_{x=0}^{x=1} x\cos\left(\frac{\pi x^2}{2}\right) dx = \frac{1}{\pi}\int_{t=0}^{t=\pi/2} \cos(t)\, dt = \frac{1}{\pi}\left[\sin(t)\right]_0^{\pi/2} = \frac{1}{\pi}.$$

(b) These are the cylindrical coordinates we have studied in the course. The Jacobian is the determinant of the following matrix

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, the element of volume which we need to use for the integral is

$$dx\, dy\, dz = |J|\, dr\, d\theta\, dz = r\, dr\, d\theta\, dz.$$

We will need to use this element of volume for the next part of the exercise. Here they ask us to compute the mass of a solid and they tell us that its density is the function $(x^2 + y^2)z$. Since the density is mass per unit of volume, what the problem is asking us is to integrate the density function in the volume bounded by the cone and the cylinder, whose equations are given in the problem. In cylindrical coordinates the density is just

$$(x^2 + y^2)z = r^2 z,$$

and the equations of the cone and the cylinder become

$$z^2 = r^2, \qquad z \geq 0,$$

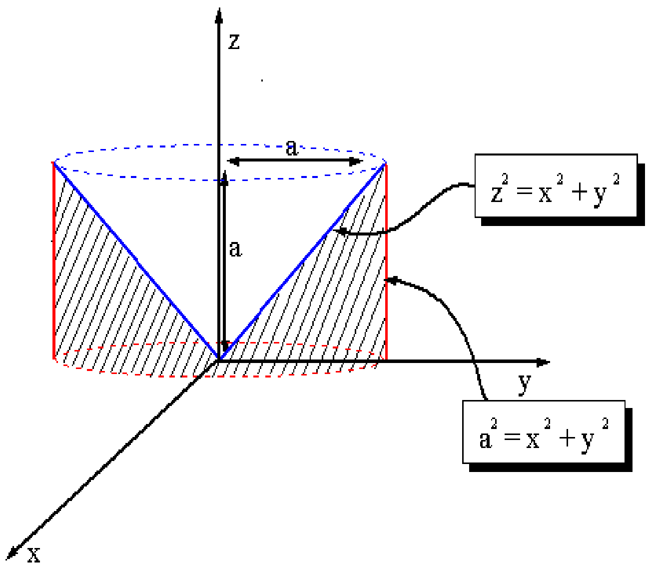
and

$$r^2 = a^2,$$

respectively. Since r is always positive (it is a distance!) the equations above are equivalent to:

$$z = r \qquad \text{and} \qquad r = a.$$

Therefore, the integration region is the dashed volume as sketched in the figure below,



and corresponds to

$$R = \{(r, \theta, z)\,|\, z \leq r \leq a, \quad 0 \leq \theta \leq 2\pi, \qquad 0 \leq z \leq a\}.$$

Therefore, we have to do the integral

$$m = \int_{z=0}^{z=a} \int_{r=z}^{r=a} \int_{\theta=0}^{\theta=2\pi} r^3 z dr\, dz\, d\theta.$$

Notice that the r^3 in the integral comes from the factor r^2 of the density function and the factor r in the Jacobian. The first integral is simply

$$\int_{\theta=0}^{\theta=2\pi} r^3 z d\theta = 2\pi r^3 z.$$

Plugging that back into the r-integral we obtain

$$\int_{r=z}^{r=a} 2\pi r^3 z dr = \left[2\pi z \frac{r^4}{4}\right]_{r=z}^{r=a} = \frac{\pi}{2} z(a^4 - z^4).$$

We can now finally compute the mass by carrying out the last integral in z

$$m = \int_{z=0}^{z=a} \frac{\pi}{2} z(a^4 - z^4) dz = \frac{\pi}{2}\left[a^4 \frac{z^2}{2} - \frac{z^6}{6}\right]_{z=0}^{z=a} = \frac{\pi}{2}\left(a^4 \frac{a^2}{2} - \frac{a^6}{6}\right) - 0 = \frac{\pi a^6}{2}\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{\pi a^6}{6}.$$

2. The Taylor expansion of a function of two variables $f(x, y)$ around a point (x_0, y_0) up to second order terms is given by

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0), \end{aligned}$$

assuming $f_{xy} = f_{yx}$. In our case $(x_0, y_0) = (1, -1)$ and

$$\begin{aligned} f_x &= (2x + x^2 - y - xy + y^2)e^{x+y}, \quad f_y = (2y - x + x^2 - xy + y^2)e^{x+y}, \\ f_{xx} &= (2 + 4x + x^2 - 2y - xy + y^2)e^{x+y}, \quad f_{yy} = (2 - 2x + x^2 + 4y - xy + y^2)e^{x+y}, \\ f_{xy} &= f_{yx} = (-1 + x + x^2 + y - xy + y^2)e^{x+y}. \end{aligned}$$

Therefore

$$\begin{aligned} f_x(1,-1) &= 6, \quad f_y(1,-1) = 0, \quad f_{xx}(1,-1) = 11, \\ f_{yy}(1,-1) &= -1, \qquad f_{xy}(1,-1) = f_{yx}(1,-1) = 2, \end{aligned}$$

and $f(1, -1) = 3$. With this we obtain the following Taylor expansion

$$\begin{aligned} f(x,y) &= 3 + 6(x-1) + \frac{11}{2}(x-1)^2 - \frac{1}{2}(y+1)^2 + 2(x-1)(y+1) \\ &= \frac{1}{2}(11x^2 - y^2 + 4xy - 6(x+y)). \end{aligned}$$

The Taylor expansion in terms of the displacements h and k is obtained simply by replacing $x = x_0 + h = h + 1$ and $y = y_0 + k = k - 1$ in our last formula. It gives

$$f(h,k) = \frac{1}{2}(6 + 12h + 4hk + 11h^2 - k^2).$$

The approximate value of $f(1.1, -0.95)$ is

$$f(1.1,-0.95) \simeq \frac{1}{2}(11(1.1)^2 - (0.95)^2 - 4(1.1)(0.95) - 6(1.1 - 0.95)) = 3.6638.$$

The exact value of the function at that point is

$$f(1.1,-0.95) = ((1.1)^2 + (0.95)^2 + (1.1)(0.95))e^{1.1-0.95} = 3.66849,$$

so the Taylor expansion up to second-order terms is a very good approximation for the point $(1.1, -0.95)$.

(b) (b) In this case our constraint is

$$\phi(x,y) = y^2 + x^2 + 4xy - 4 = 0, \tag{0.1}$$

and the function we want to minimize is the distance from the point $(0,0)$ to a point (x,y) in the curve above. The square of the distance is the function

$$f(x,y) = x^2 + y^2,$$

and the key thing to notice in the problem is that the curve lies on the xy-plane and therefore the point which is closest to $(0,0)$ and lies in the curve (0.1) has coordinate $z=0$ (it is contained on the xy-plane). This means that we have a problem of Lagrange multipliers but we only have equations in x and y. The corresponding partial derivatives of f and ϕ are

$$\begin{aligned} f_x &= 2x, \qquad f_y = 2y, \\ \phi_x &= 2x+4y, \qquad \phi_y = 2y+4x. \end{aligned}$$

Therefore we need to solve the following system of equations

$$\begin{aligned} y^2 + x^2 + 4xy - 4 = 0 &= 0, \\ 2x + \lambda(2x+4y) &= 0, \\ 2y + \lambda(2y+4x) &= 0. \end{aligned}$$

The last two equations give

$$\lambda = -\frac{x}{x+2y} = -\frac{y}{y+2x}.$$

and from this equality we obtain

$$x(y+2x) = y(x+2y) \quad \Rightarrow \quad x^2 = y^2 \quad \Rightarrow \quad x = \pm y.$$

For $x=y$ we obtain $\lambda = -1/2$ and for $x=-y$ we have $\lambda = 1$. Substituting $x=y$ into the constraint (0.1) we obtain

$$6x^2 - 4 = 0 \quad \Rightarrow \quad x^2 = \frac{2}{3} \quad \Rightarrow \quad x = y = \pm\sqrt{\frac{2}{3}}.$$

Taking now the other solution $x=-y$ and substituting it into (0.1) we obtain

$$-2x^2 - 4 = 0 \quad \Rightarrow \quad x^2 = -2,$$

and this solution does not make sense, since it gives x,y imaginary. Therefore the only sensible solutions to the problem are

$$x = y = \sqrt{\frac{2}{3}}, \qquad x = y = -\sqrt{\frac{2}{3}}.$$

and substituting them into the square distance $f(x,y)$ we see that they give us the same distance

$$d = \sqrt{f(\pm\sqrt{\frac{2}{3}}, \pm\sqrt{\frac{2}{3}})} = \sqrt{\frac{4}{3}}.$$

Therefore there are two points contained in the curve (0.1) which are both at the same distance from $(0,0,0)$ and this is also the shortest distance.

3. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition

$$m^2 - 1 = 0 \Rightarrow m = \pm 1.$$

This means that the general solution of the homogeneous equation is of the form

$$y = c_1 e^x + c_2 e^{-x},$$

therefore we identify

$$u_1(x) = e^x, \qquad u_2(x) = e^{-x}.$$

For the second part of the problem we will need the Wronskian of these solutions which is

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2.$$

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form

$$y = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx.$$

In our case

$$R(x) = \frac{2}{e^{-x} + e^x}, \qquad W(x) = -2,$$

therefore

$$v_1(x) = \int \frac{e^{-x}}{e^{-x} + e^x}dx = \int \frac{dx}{1 + e^{2x}} = \int \frac{e^{-2x}}{1 + e^{-2x}}\,dx = -\frac{1}{2}\ln|1 + e^{-2x}|.$$

$$v_2(x) = -\int \frac{e^x}{e^{-x} + e^x}dx = -\int \frac{dx}{1 + e^{-2x}} = -\int \frac{e^{2x}}{1 + e^{2x}}\,dx = -\frac{1}{2}\ln|1 + e^{2x}|.$$

Hence the general solution of the inhomogeneous equation is

$$y = c_1 e^x + c_2 e^{-x} - \frac{e^x}{2}\ln|1 + e^{-2x}| - \frac{e^{-x}}{2}\ln|1 + e^{2x}|.$$

with c_1, c_2 being arbitrary constants.

4. (a) Here we can use the chain rule

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt},$$

for a function of two variables $f(x,y)$ under a change of coordinates which relates the original variables x,y to a **single** variable t. In this particular case we can compute

$$\frac{dx}{dt} = -\sin(t), \qquad \frac{dy}{dt} = 2\cos(t), \qquad \frac{\partial f}{\partial x} = -2xe^{-(x^2+y^2)} \quad \text{and} \quad \frac{\partial f}{\partial y} = -2ye^{-(x^2+y^2)},$$

and therefore

$$\frac{df}{dt} = 2\sin(t)xe^{-(x^2+y^2)} - 4\cos(t)ye^{-(x^2+y^2)} = 2e^{-(x^2+y^2)}\left(\sin(t)x - 2\cos(t)y\right).$$

The problem tells us to express df/dt in terms of t, so to finish the problem we have to substitute all the x and y in terms of t in the previous formula

$$\frac{df}{dt} = 2e^{-(\cos^2(t)+4\sin^2(t))}\left(\sin(t)\cos(t) - 4\cos(t)\sin(t)\right) = -6\sin(t)\cos(t)e^{-(1+3\sin^2(t))},$$

were we used $\sin^2(t) + \cos^2(t) = 1$.

An alternative (and shorter) way of doing the problem is to substitute $x = \cos(t)$ and $y = 2\sin(t)$ directly into the function $f(x,y)$. That gives us

$$f(x(t), y(t)) = e^{-(1+3\sin^2(t))},$$

and then do the derivative

$$\frac{df}{dt} = -6\sin(t)\cos(t)e^{-(1+3\sin^2(t))}.$$

(b) Here we need to use the chain rule for a function of two variables x,y which are changed to two new variables u,v. The relevant identities are

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \tag{0.2}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}. \tag{0.3}$$

For $x = (u^2 - v^2)/2$ and $y = uv$ we have

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} = u, \qquad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u} = -v.$$

Plugging these derivatives into (0.2)-(0.3) we obtain the formulae given in the problem. The second order partial derivatives are obtained from these formulae as

$$\begin{aligned}
\frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial u}(uf_x + vf_y) = f_x + u\frac{\partial f_x}{\partial u} + v\frac{\partial f_y}{\partial u} \\
&= f_x + u(uf_{xx} + vf_{yx}) + v(uf_{xy} + vf_{yy}) = f_x + u^2 f_{xx} + v^2 f_{yy} + 2uvf_{xy}, \\
\frac{\partial^2 f}{\partial v^2} &= \frac{\partial}{\partial v}(-vf_x + uf_y) = -f_x - v\frac{\partial f_x}{\partial v} + u\frac{\partial f_y}{\partial v} \\
&= -f_x - v(-vf_{xx} + uf_{yx}) + u(-vf_{xy} + uf_{yy}) = -f_x + v^2 f_{xx} + u^2 f_{yy} - 2uvf_{xy},
\end{aligned}$$

and

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = (u^2 + v^2)(f_{xx} + f_{yy}).$$

Therefore if $f_{xx} + f_{yy} = 0$, then automatically $f_{uu} + f_{vv} = 0$.