CALCULUS 2003: EXAM SOLUTIONS

1. (a) The integration region is the triangle formed by the intersection of the lines $x = y$, $y = 0$ and $x = 1$. Changing the order of integration we obtain the integral

$$I = \int_{x=0}^{x=1} dx \int_{y=0}^{y=x} \frac{e^x}{x} dy.$$

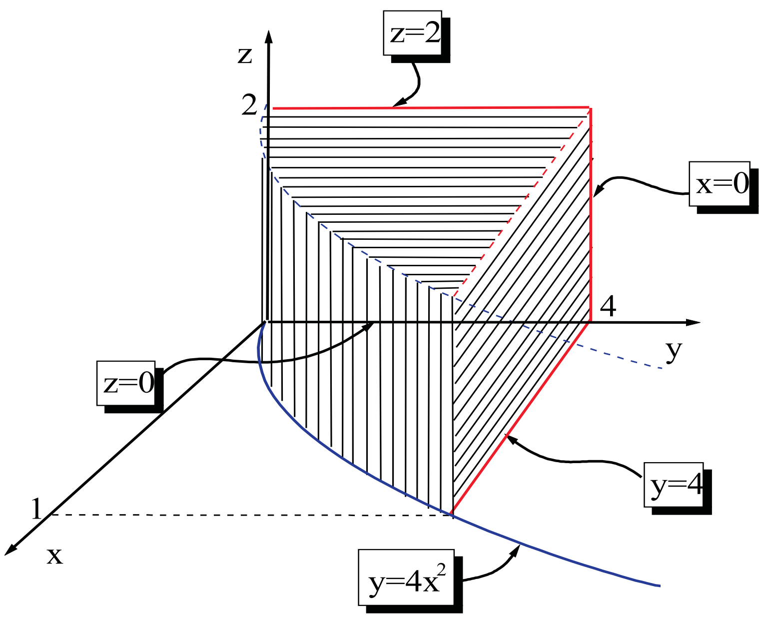
The integral in y gives

$$\int_{y=0}^{y=x} \frac{e^x}{x} dy = \left[y\frac{e^x}{x} \right]_0^x = e^x.$$

Plugging this result into the second integral we obtain

$$I = \int_{x=0}^{x=1} e^x dx = [e^x]_0^1 = e - 1.$$

(b) As usual we start by sketching the integration region



From the picture and the information given by the problem we can deduce that the integration region is

$$R = \{(x, y, z) : 0 \leq x \leq 1, \quad 4x^2 \leq y \leq 4, \quad 0 \leq z \leq 2.\}$$

The integral is

$$I = 2\int_{x=0}^{x=1} x dx \int_{y=4x^2}^{4} dy \int_{z=0}^{2} dz.$$

The integral in z is

$$\int_{z=0}^{2} dz = [z]_0^2 = 2.$$

The integral in y is

$$\int_{y=4x^2}^{4} dy = [y]_{4x^2}^{4} = 4(1-x^2).$$

Therefore, the final result is

$$I = 16\int_{x=0}^{x=1} x(1-x^2)dx = 16\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1 = 16\left(\frac{1}{2}-\frac{1}{4}\right) = 4.$$

2. (a) The first order partial derivatives are

$$\begin{aligned} f_x &= 3x^2+y^2-24x+21, \\ f_y &= 2xy-4y. \end{aligned}$$

The first thing we have to do is finding the points at which these derivatives vanish

$$f_y = 0 \quad \Rightarrow \; y(x-2)=0 \quad \Rightarrow y=0 \;\text{ or }\; x=2.$$

For $y=0$ (which is one of the solutions of the previous equation) f_x will vanish if

$$f_x(x,y=0) = 0 = 3x^2-24x+21=0 \;\Rightarrow\; x = \frac{24\pm 18}{6} = 7, 1,$$

and for $x=2$ (which is the other solution of $f_y=0$) we would obtain

$$f_x(x=2,y)=0 = 12+y^2-48+21 \quad \Rightarrow \quad y = \pm\sqrt{15}.$$

Therefore, putting all these solutions together we have the following 4 points:

$$(x,y) = (1,0),\; (7,0),\; (2,\sqrt{15}) \quad\text{and}\quad (2,-\sqrt{15}).$$

The next step is to compute the second order partial derivatives

$$\begin{aligned} A &= f_{xx} = 6x-24, \\ B &= f_{xy} = f_{yx} = 2y, \\ C &= f_{yy} = 2x-4, \end{aligned}$$

therefore

$$AC-B^2 = (6x-24)(2x-4)-4y^2,$$

and we have to study the sign of this quantity in order to classify the stationary points of the function:

The point $(1,0)$**:** At this point

$$\begin{aligned} AC-B^2 &= (6-24)(2-4) = 36 > 0, \\ A &= 6-24 = -18 < 0, \end{aligned}$$

therefore this point is a **maximum**.

The point $(7,0)$**:** At this point

$$\begin{aligned} AC - B^2 &= (42-24)(14-4) = 180 > 0, \\ A &= 42 - 24 = 18 > 0, \end{aligned}$$

therefore this point is a **minimum**.

The point $(2,\sqrt{15})$**:** At this point

$$AC - B^2 = (12-24)(4-4) - 60 = -60 < 0,$$

therefore this point is a **saddle point**.

The point $(2,-\sqrt{15})$**:** At this point

$$AC - B^2 = (12-24)(4-4) - 60 = -60 < 0,$$

therefore this point is also a **saddle point**.

(b) The Taylor expansion of a function of two variables $f(x,y)$ around a point (x_0,y_0) up to second order terms is given by

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0), \end{aligned}$$

assuming $f_{xy} = f_{yx}$. In our case $(x_0,y_0) = (-1,-1)$ and

$$\begin{aligned} f_x &= (1+x+y)e^{x-y}, \quad f_y = (1-x-y)e^{x-y}, \quad f_{xx} = (2+x+y)e^{x-y}, \\ f_{yy} &= (-2+x+y)e^{x-y}, \qquad f_{xy} = f_{yx} = -(x+y)e^{x-y}. \end{aligned}$$

Therefore

$$\begin{aligned} f_x(-1,-1) &= -1, \quad f_y(-1,-1) = 3, \quad f_{xx}(-1,-1) = 0, \\ f_{yy}(-1,-1) &= -4, \qquad f_{xy}(-1,-1) = f_{yx}(-1,-1) = 2, \end{aligned}$$

and $f(-1,-1) = -2$. With this we obtain the following Taylor expansion

$$\begin{aligned} f(x,y) &= -2 - (x+1) + 3(y+1) - 2(y+1)^2 + 2(y+1)(x+1) \\ &= y + x - 2y^2 + 2xy. \end{aligned}$$

Therefore, the approximate value of $f(-0.9,-1.05)$ is

$$f(-0.9,-1.05) \simeq -0.9 - 1.05 - 2(1.05)^2 + 2(0.9)(1.05) = -2.265.$$

The Taylor expansion in terms of the displacements h and k is obtained simply by replacing $x = x_0 + h = h - 1$ and $y = y_0 + k = k - 1$ in our final formula. It gives

$$f(h,k) = k + h - 2 - 2(k-1)^2 + 2(k-1)(h-1).$$

3. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition

$$m^2 - 1 = 0 \Rightarrow m = \pm 1.$$

This means that the general solution of the homogeneous equation is of the form

$$y = c_1 e^x + c_2 e^{-x},$$

therefore we identify

$$u_1(x) = e^x, \qquad u_2(x) = e^{-x}.$$

For the second part of the problem we will need the Wronskian of these solutions which is

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2.$$

Therefore the Wronskian is indeed nowhere zero.

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form

$$y = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx.$$

In our case

$$R(x) = \frac{1}{1+e^x}, \qquad W(x) = -2,$$

therefore

$$\begin{aligned} v_1(x) &= \frac{1}{2}\int \frac{e^{-x}}{1+e^x}dx = \frac{1}{2}\int \frac{1}{e^x(1+e^x)}dx = \frac{1}{2}\int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)dx \\ &= -\frac{e^{-x}}{2} - \frac{1}{2}\int \left(\frac{1+e^x-e^x}{1+e^x}\right)dx = -\frac{e^{-x}}{2} - \frac{1}{2}\int \left(1 - \frac{e^x}{1+e^x}\right)dx \\ &= -\frac{e^{-x}}{2} - \frac{x}{2} + \frac{1}{2}\ln|1+e^x|. \end{aligned}$$

$$v_2(x) = -\frac{1}{2}\int \frac{e^x}{1+e^x}dx = -\frac{1}{2}\ln|1+e^x|.$$

Hence the general solution of the inhomogeneous equation is

$$y = c_1 e^x + c_2 e^{-x} + e^x\left(-\frac{e^{-x}}{2} - \frac{x}{2} + \frac{1}{2}\ln|1+e^x|\right) - e^{-x}\frac{1}{2}\ln|1+e^x|.$$

with c_1, c_2 being arbitrary constants.

4. (a) Here we simply have to use the chain rule

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta f_x + \sin\theta f_y,$$

and

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta f_x + r\cos\theta f_y.$$

In order to prove the identity we just need to compute employing the formulae above for $f = V$

$$V_r^2 + \frac{1}{r^2}V_\theta^2 = (\cos\theta V_x + \sin\theta V_y)^2 + \frac{1}{r^2}(-r\sin\theta V_x + r\cos\theta V_y)^2 = V_x^2 + V_y^2.$$

(b) Let us consider an implicit function of two variables $z = f(x, y)$ and assume the existence of a constraint

$$F(x, y, z) = 0,$$

which relates the function z to the two independent variables x and y. Since $F = 0$ it is clear that also its total differential $dF = 0$ must vanish. However the total differential is by definition

$$dF = \left(\frac{\partial F}{\partial x}\right) dx + \left(\frac{\partial F}{\partial y}\right) dy + \left(\frac{\partial F}{\partial z}\right) dz = 0, \tag{0.1}$$

and in addition, z is a function of x and y, therefore its differential is given by

$$dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy. \tag{0.2}$$

If we substitute (0.2) into (0.1) we obtain the equation

$$dF = 0 = \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\right) dy. \tag{0.3}$$

Since x and y are independent variables, equation (0.3) implies that each of the factors has to vanish separately, that is

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0.$$

Therefore we obtain,

$$\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)} = -\frac{F_x}{F_z},$$

$$\frac{\partial z}{\partial y} = -\frac{\left(\frac{\partial F}{\partial y}\right)}{\left(\frac{\partial F}{\partial z}\right)} = -\frac{F_y}{F_z}.$$

Employing now these formulae for the function $F(x, y, z) = z \tan x - xy^2z^3 - 2xyz = 0$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\frac{z}{\cos^2 x} - y^2z^3 - 2yz}{\tan x - 3xy^2z^2 - 2xy},$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{2yz^3 + 2xz}{\tan x - 3xy^2z^2 - 2xy}.$$