CALCULUS 2005: EXAM SOLUTIONS

1. (a) The integration region is the triangle formed by the intersection of the lines $y = x$, $y = 0$ and $x = 1$. Once we have identified the integration region, it is easy to change the order of integration to write I equivalently as

$$I = \int_{x=0}^{x=1} dx \int_{y=0}^{y=x} \cos\left(\frac{\pi x^2}{2}\right) dy.$$

The integral

$$\int_{y=0}^{y=x} \cos\left(\frac{\pi x^2}{2}\right) dy = \left[y \cos\left(\frac{\pi x^2}{2}\right)\right]_0^x = x \cos\left(\frac{\pi x^2}{2}\right) - 0 = x \cos\left(\frac{\pi x^2}{2}\right),$$

is trivial to do, since the argument does not depend on y. Now the second integral is also very easy to do, since we have the product of the cosine of a function and the derivative of that function, therefore

$$I = \int_{x=0}^{x=1} x \cos\left(\frac{\pi x^2}{2}\right) dx = \left[\frac{1}{\pi} \sin\left(\frac{\pi x^2}{2}\right)\right]_0^1 = \frac{1}{\pi} - 0 = \frac{1}{\pi}.$$

If you do not realize how to do the integral directly, you can also change variables to $t = \pi x^2/2$ which gives $dt = \pi x dx$ and allows you to rewrite the integral above as

$$I = \int_{x=0}^{x=1} x \cos\left(\frac{\pi x^2}{2}\right) dx = \frac{1}{\pi} \int_{t=0}^{t=\pi/2} \cos(t)\, dt = \frac{1}{\pi} \left[\sin(t)\right]_0^{\pi/2} = \frac{1}{\pi}.$$

(b) These are the cylindrical coordinates we have studied in the course. The Jacobian is the determinant of the following matrix

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, the element of volume which we need to use for the integral is

$$dx\, dy\, dz = |J|\, dr\, d\theta\, dz = r\, dr\, d\theta\, dz.$$

We will need to use this element of volume for the next part of the exercise. Here they ask us to compute the mass of a solid and they tell us that its density is the function $(x^2 + y^2)z$. Since the density is mass per unit of volume, what the problem is asking us is to integrate the density function in the volume bounded by the cone and the cylinder, whose equations are given in the problem. In cylindrical coordinates the density is just

$$(x^2 + y^2)z = r^2 z,$$

and the equations of the cone and the cylinder become

$$z^2 = r^2, \qquad z \geq 0,$$

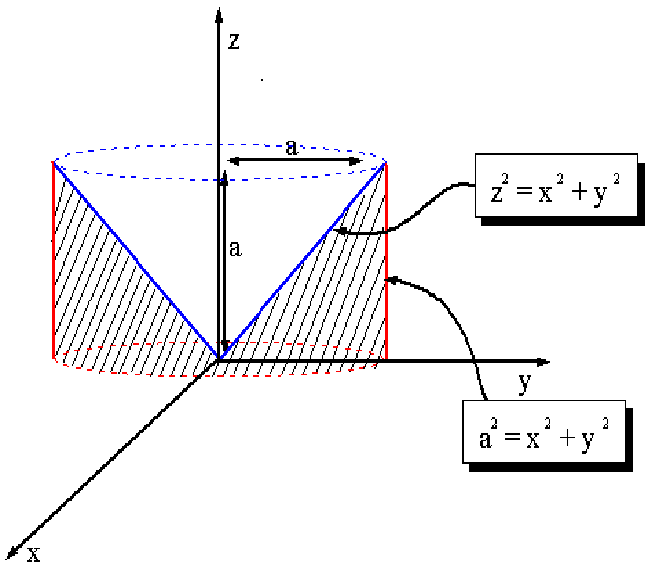
and

$$r^2 = a^2,$$

respectively. Since r is always positive (it is a distance!) the equations above are equivalent to:

$$z = r \qquad \text{and} \qquad r = a.$$

Therefore, the integration region is the dashed volume as sketched in the figure below,



and corresponds to

$$R = \{(r, \theta, z) \,|\, z \leq r \leq a, \quad 0 \leq \theta \leq 2\pi, \qquad 0 \leq z \leq a\}.$$

Therefore, we have to do the integral

$$m = \int_{z=0}^{z=a} \int_{r=z}^{r=a} \int_{\theta=0}^{\theta=2\pi} r^3 z dr\, dz\, d\theta.$$

Notice that the r^3 in the integral comes from the factor r^2 of the density function and the factor r in the Jacobian. The first integral is simply

$$\int_{\theta=0}^{\theta=2\pi} r^3 z d\theta = 2\pi r^3 z.$$

Plugging that back into the r-integral we obtain

$$\int_{r=z}^{r=a} 2\pi r^3 z dr = \left[2\pi z \frac{r^4}{4}\right]_{r=z}^{r=a} = \frac{\pi}{2} z(a^4 - z^4).$$

We can now finally compute the mass by carrying out the last integral in z

$$m = \int_{z=0}^{z=a} \frac{\pi}{2} z(a^4 - z^4) dz = \frac{\pi}{2}\left[a^4 \frac{z^2}{2} - \frac{z^6}{6}\right]_{z=0}^{z=a} = \frac{\pi}{2}\left(a^4 \frac{a^2}{2} - \frac{a^6}{6}\right) - 0 = \frac{\pi a^6}{2}\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{\pi a^6}{6}.$$

2. (a) Here we can use the chain rule

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt},$$

for a function of two variables $f(x, y)$ under a change of coordinates which relates the original variables x, y to a **single** variable t. In this particular case we can compute

$$\frac{dx}{dt} = -\sin(t), \qquad \frac{dy}{dt} = 2\cos(t), \qquad \frac{\partial f}{\partial x} = -2xe^{-(x^2+y^2)} \quad \text{and} \quad \frac{\partial f}{\partial y} = -2ye^{-(x^2+y^2)},$$

and therefore

$$\frac{df}{dt} = 2\sin(t)xe^{-(x^2+y^2)} - 4\cos(t)ye^{-(x^2+y^2)} = 2e^{-(x^2+y^2)}\left(\sin(t)x - 2\cos(t)y\right).$$

The problem tells us to express df/dt in terms of t, so to finish the problem we have to substitute all the x and y in terms of t in the previous formula

$$\frac{df}{dt} = 2e^{-(\cos^2(t)+4\sin^2(t))}\left(\sin(t)\cos(t) - 4\cos(t)\sin(t)\right) = -6\sin(t)\cos(t)e^{-(1+3\sin^2(t))},$$

were we used $\sin^2(t) + \cos^2(t) = 1$.

An alternative (and shorter) way of doing the problem is to substitute $x = \cos(t)$ and $y = 2\sin(t)$ directly into the function $f(x, y)$. That gives us

$$f(x(t), y(t)) = e^{-(1+3\sin^2(t))},$$

and then do the derivative

$$\frac{df}{dt} = -6\sin(t)\cos(t)e^{-(1+3\sin^2(t))}.$$

(b) Here we need to use the chain rule for a function of two variables x, y which are changed to two new variables u, v. The relevant identities are

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \tag{0.1}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}. \tag{0.2}$$

It is easy to realize that **the problem is wrongly formulated**, as it is **not** possible to obtain

$$\frac{\partial f}{\partial u} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}, \tag{0.3}$$

$$\frac{\partial f}{\partial v} = -v\frac{\partial f}{\partial x} + u\frac{\partial f}{\partial y}, \tag{0.4}$$

from the relations $x = (u+v)/2$ and $y = (u^2+v^2)/4$. In order to obtain (0.3)-(0.4) **the correct transformation of coordinates is**

$$x = \frac{u^2 - v^2}{2}, \qquad y = uv,$$

which implies

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} = u, \qquad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u} = -v.$$

Plugging these derivatives into (0.1)-(0.2) we obtain (0.3)-(0.4). The second order partial derivatives are obtained from (0.3)-(0.4) as

$$\begin{aligned}
\frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial u}(uf_x + vf_y) = f_x + u\frac{\partial f_x}{\partial u} + v\frac{\partial f_y}{\partial u} \\
&= f_x + u(uf_{xx} + vf_{yx}) + v(uf_{xy} + vf_{yy}) = f_x + u^2 f_{xx} + v^2 f_{yy} + 2uvf_{xy}, \\
\frac{\partial^2 f}{\partial v^2} &= \frac{\partial}{\partial v}(-vf_x + uf_y) = -f_x - v\frac{\partial f_x}{\partial v} + u\frac{\partial f_y}{\partial v} \\
&= -f_x - v(-vf_{xx} + uf_{yx}) + u(-vf_{xy} + uf_{yy}) = -f_x + v^2 f_{xx} + u^2 f_{yy} - 2uvf_{xy},
\end{aligned}$$

and

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = (u^2 + v^2)(f_{xx} + f_{yy}).$$

Therefore if $f_{xx} + f_{yy} = 0$, then automatically $f_{uu} + f_{vv} = 0$.

3. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition

$$m^2 - 4m + 5 = 0 \Rightarrow m = 2 \pm i.$$

This means that the general solution of the homogeneous equation is of the form

$$y = c_1 e^{2x}\cos x + c_2 e^{2x}\sin x,$$

therefore we identify

$$u_1(x) = e^{2x}\cos x, \qquad u_2(x) = e^{2x}\sin x.$$

For the second part of the problem we will need the Wronskian of these solutions which is

$$\begin{aligned}
W(x) &= \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} e^{2x}\cos x & e^{2x}\sin x \\ 2e^{2x}\cos x - e^{2x}\sin x & 2e^{2x}\sin x + e^{2x}\cos x \end{vmatrix} \\
&= e^{4x}\cos x(2\sin x + \cos x) - e^{4x}\sin x(2\cos x - \sin x) = e^{4x}.
\end{aligned}$$

Therefore the Wronskian is indeed nowhere zero for finite values of x.

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form

$$y = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx.$$

In our case

$$R(x) = \frac{e^{2x}}{\sin x}, \qquad W(x) = e^{4x},$$

therefore

$$\begin{aligned} v_1(x) &= -\int dx = -x, \\ v_2(x) &= \int \frac{\cos x}{\sin x}dx = \ln|\sin x|. \end{aligned}$$

Hence the general solution of the inhomogeneous equation is

$$y = e^{2x}(c_1 \cos x + c_2 \sin x - x\cos x + \ln|\sin x|\sin x),$$

with c_1, c_2 being arbitrary constants.

4. (a) The Taylor expansion of a function of two variables $f(x, y)$ around a point (x_0, y_0) up to second order terms is given by

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0). \end{aligned}$$

First we need to compute the 1st and 2nd order partial derivatives

$$\begin{aligned} f_x &= 2e^{2x+3y}\left(8x + 8x^2 - 3y - 6xy + 3y^2\right), \\ f_y &= 3e^{2x+3y}\left(-2x + 8x^2 + 2y - 6xy + 3y^2\right), \\ f_{xx} &= 4e^{2x+3y}\left(4 + 16x + 8x^2 - 6y - 6xy + 3y^2\right), \\ f_{yy} &= 3e^{2x+3y}\left(2 - 12x + 24x^2 + 12y - 18xy + 9y^2\right), \\ f_{xy} &= f_{yx} = 6e^{2x+3y}\left(-1 + 6x + 8x^2 - y - 6xy + 3y^2\right). \end{aligned}$$

Therefore

$$\begin{aligned} f_x(0,0) &= 0, \quad f_y(0,0) = 0, \quad f_{xx}(0,0) = 16, \\ f_{yy}(0,0) &= 6, \qquad f_{xy}(0,0) = f_{yx}(0,0) = -6, \end{aligned}$$

and $f(0,0) = 0$. With this we obtain the following Taylor expansion

$$f(x,y) = 8x^2 + 3y^2 - 6xy.$$

To obtain the expansion in terms of the displacements h and k we only need to set $x = x_0 + h$ and $y = y_0 + k$. Since in this case $x_0 = y_0 = 0$,

$$f(h,k) = 8h^2 + 3k^2 - 6hk.$$

The problem also asks what we can conclude about the nature of the point $(0, 0)$. Since both first order derivatives vanish at that point we know that it must be either a maximum, a minimum or a saddle point. To know which one it is we need to compute:

$$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = (16)(6) - 6^2 = 60 > 0.$$

Since $f_{xx}(0,0) = 16 > 0$ the point is in fact a minimum of the function.

(b) In this case our constraint is

$$\phi(x,y,z) = x^3 + y^3 + z^3 - 1 = 0, \tag{0.5}$$

and the corresponding partial derivatives of f and ϕ are

$$\begin{array}{rcl} f_x & = & zy, \qquad f_y = xz, \qquad f_z = xy, \\ \phi_x & = & 3x^2, \qquad \phi_y = 3y^2, \qquad \phi_z = 3z^2. \end{array}$$

Therefore we need to solve the following system of equations

$$\begin{array}{rcl} x^3 + y^3 + z^3 - 1 & = & 0, \\ zy + \lambda 3x^2 & = & 0, \\ zx + \lambda 3y^2 & = & 0, \\ xy + \lambda 3z^2 & = & 0. \end{array}$$

The last three equations are solved by $x = y = z$ and $\lambda = -1/3$, which when plugged into the first equation gives the condition

$$3x^3 = 1 \Rightarrow x = \sqrt[3]{\frac{1}{3}}.$$

In addition, the equations admit also the solutions $(0,0,1)$, $(1,0,0)$ and $(0,1,0)$ with $\lambda = 0$. At these points $f = 0$ and this is the minimum value of this function for points satisfying (0.5) and $x, y, z \geq 0$. The maximum value of f subject to (0.5) is therefore $1/3$.