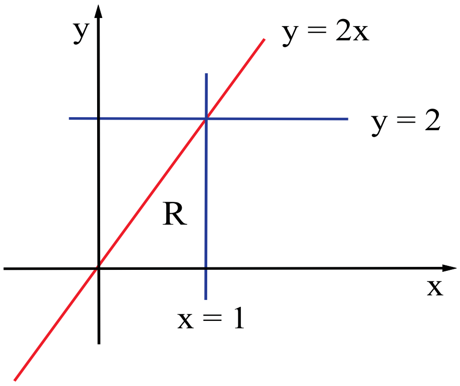
CALCULUS 2006: EXAM SOLUTIONS

1. (a) The integration region is the lower triangle in the picture [3]



From the picture it is easy to see that changing the order of integration we obtain [3]

$$I = \int_{x=0}^{x=1} dx \int_{y=0}^{y=2x} \cos(x^2) dy.$$

The integral in y gives [2]

$$\int_{y=0}^{y=2x} \cos(x^2) dy = \left[y \cos(x^2)\right]_0^{2x} = 2x \cos(x^2).$$

Plugging this result into the second integral we obtain [2]

$$I = \int_{x=0}^{x=1} 2x \cos(x^2) dx = \left[\sin(x^2)\right]_0^1 = \sin(1).$$

(b) The Jacobian of the change of coordinates is simply [2]

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

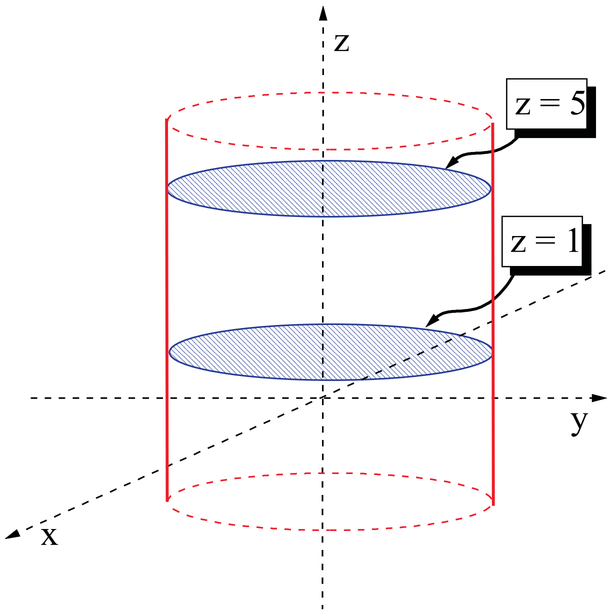
Therefore, the element of volume which we need to use for the integral is [1]

$$dx\, dy\, dz = |J|\, dr\, d\theta\, dz = r\, dr\, d\theta\, dz.$$

To compute the integral we have first to express the integrand in terms of the new variables, that is [1]

$$(x^2 + y^2)^2 = (r^2)^2 = r^4.$$

The next step is to describe the region of integration in terms of the new variables. The integration region for this problem is very easy to sketch. We have a radius 1 circular cylinder centered at the origin extending between the $z = 1$ and $z = 5$ planes. This looks more or less like in the picture below



In cylindrical coordinates, the integration region is simply 2

$$R = \{(r, z, \theta) : 0 \le r \le 1, \quad 1 \le z \le 5, \quad 0 \le \theta \le 2\pi\},$$

and the integral we want to compute is therefore 1

$$V = \int_{r=0}^{r=1} r^5 dr \int_{\theta=0}^{\theta=2\pi} d\theta \int_{z=1}^{z=5} dz.$$

The various integrals can be carried out separately and give 2

$$\int_{r=0}^{r=1} r^5 dr = \left[\frac{r^6}{6}\right]_0^1 = \frac{1}{6}, \quad \int_{\theta=0}^{\theta=2\pi} d\theta = 2\pi, \quad \int_{z=1}^{z=5} dz = 5 - 1 = 4.$$

Therefore 1

$$V = (2\pi)(1/6)(4) = \frac{4\pi}{3}.$$

2. (a) First of all we need to find the points at which the first order partial derivatives vanish. These derivatives are 2

$$f_x = (1 - 2x^2)e^{-x^2+y^2}, \qquad f_y = 2xye^{-x^2+y^2}.$$

Then 1

$$f_x = 0 \quad \Leftrightarrow \quad x = \pm\frac{1}{\sqrt{2}},$$

and [1]

$$f_y = 0 \quad \Leftrightarrow \quad x = 0 \quad \texttt{or} \quad y = 0.$$

That gives us 2 candidates to be stationary points, that is the points $(\pm 1/\sqrt{2}, 0)$ at which both f_x and f_y vanish. To investigate what type of stationary points this points are, we have to look at the second order partial derivatives: [2]

$$f_{xx} = 2x(-3 + 2x^2)e^{-x^2+y^2}, \qquad f_{yy} = 2x(1 + 2y^2)e^{-x^2+y^2},$$

$$f_{xy} = f_{yx} = 2y(1 - 2x^2)e^{-x^2+y^2}.$$

Calling $A = f_{xx}$, $B = f_{xy}$ and $C = f_{yy}$, we find:

i) For the point $(1/\sqrt{2}, 0)$ we have [2]

$$A = -2\sqrt{\frac{2}{e}}, \qquad B = 0, \qquad C = \sqrt{\frac{2}{e}}.$$

Then

$$AC - B^2 = -\frac{4}{e} < 0,$$

therefore this point is a saddle point.

ii) For the point $(-1/\sqrt{2}, 0)$ we have [2]

$$A = 2\sqrt{\frac{2}{e}}, \qquad B = 0, \qquad C = -\sqrt{\frac{2}{e}}.$$

Then

$$AC - B^2 = -\frac{4}{e} < 0,$$

therefore this point is also a saddle point.

(b) The Taylor expansion of a function of two variables $f(x, y)$ around a point (x_0, y_0) up to second order terms is given by [2]

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0), \end{aligned}$$

assuming $f_{xy} = f_{yx}$. In our case $(x_0, y_0) = (0, 1)$ and [2]

$$f_x = 2x + y, \quad f_y = 3y^2 + x, \quad f_{xx} = 2, \quad f_{yy} = 6y, \quad f_{xy} = f_{yx} = 1.$$

Therefore [2]

$$f(0,1) = 1, \quad f_x(0,1) = 1, \quad f_y(0,1) = 3, \quad f_{xx}(0,1) = 2, \quad f_{yy}(0,1) = 6, \quad f_{xy}(0,1) = f_{yx}(0,1) = 1.$$

So, the Taylor expansion is [2]

$$\begin{aligned} f(x,y) &= 1 + x + 3(y-1) + x^2 + 3(y-1)^2 + x(y-1), \\ &= 1 + x^2 + xy + 3y(y-1), \end{aligned}$$

and $\boxed{2}$

$$f(0.1, 1.1) = 1 + (0.1)^2 + (0.1)(1.1) + 3(1.1)(0.1) = 1 + 0.01 + 0.44 = 1.45.$$

The exact value of the function at this point is

$$f(0.1, 1.1) = (0.1)^2 + (0.1)(1.1) + (1.1)^3 = 1.451,$$

therefore the Taylor approximation is in fact very good for this point!

3. (a) Using the chain rule we have $\boxed{4}$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = e^{u+v} f_x + e^{u-v} f_y,$$

and $\boxed{4}$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = e^{u+v} f_x - e^{u-v} f_y.$$

(b) From (a) we can obtain the 2nd order partial derivatives by using once more the chain rule we have $\boxed{5}$

$$\begin{aligned} \frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial u}(e^{u+v} f_x + e^{u-v} f_y) = e^{u+v} f_x + e^{u+v}\frac{\partial f_x}{\partial u} + e^{u-v} f_y + e^{u-v}\frac{\partial f_y}{\partial u} \\ &= e^{u+v} f_x + e^{u-v} f_y + e^{u+v}(e^{u+v} f_{xx} + e^{u-v} f_{yx}) + e^{u-v}(e^{u+v} f_{xy} + e^{u-v} f_{yy}) \\ &= e^{u+v} f_x + e^{u-v} f_y + e^{2(u+v)} f_{xx} + e^{2(u-v)} f_{yy} + e^{2u}(f_{xy} + f_{yx}), \end{aligned}$$

and $\boxed{5}$

$$\begin{aligned} \frac{\partial^2 f}{\partial v^2} &= \frac{\partial}{\partial v}(e^{u+v} f_x - e^{u-v} f_y) = e^{u+v} f_x + e^{u+v}\frac{\partial f_x}{\partial v} + e^{u-v} f_y - e^{u-v}\frac{\partial f_y}{\partial v} \\ &= e^{u+v} f_x + e^{u-v} f_y + e^{u+v}(e^{u+v} f_{xx} - e^{u-v} f_{yx}) - e^{u-v}(e^{u+v} f_{xy} - e^{u-v} f_{yy}) \\ &= e^{u+v} f_x + e^{u-v} f_y + e^{2(u+v)} f_{xx} + e^{2(u-v)} f_{yy} - e^{2u}(f_{xy} + f_{yx}). \end{aligned}$$

Substracting the two formulae we trivially see that $\boxed{2}$

$$\frac{\partial^2 f}{\partial u^2} - \frac{\partial^2 f}{\partial v^2} = 2e^{2u}\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial x}\right).$$

4. To obtain the general solution of the homogeneous equation we try solutions of the type $y = ce^{mx}$. Substituting this solution into the equation we obtain the condition $\boxed{2}$

$$m^2 - 4 = 0 \Rightarrow m = \pm 2.$$

This means that the general solution of the homogeneous equation is of the form $\boxed{2}$

$$y = c_1 e^{2x} + c_2 e^{-2x},$$

therefore we identify

$$u_1(x) = e^{2x}, \qquad u_2(x) = e^{-2x}.$$

For the second part of the problem we will need the Wronskian of these solutions which is 3

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4.$$

Therefore the Wronskian is indeed nowhere zero.

The method of variation of parameters tells us that a particular solution of the inhomogeneous equation is of the form 3

$$y = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx.$$

In our case

$$R(x) = \cosh(2x), \qquad W(x) = -4,$$

therefore 4

$$\begin{aligned} v_1(x) &= \frac{1}{4}\int e^{-2x}\cosh(2x)dx = \frac{1}{4}\int \frac{e^{-2x}(e^{2x}+e^{-2x})}{2}dx \\ &= \frac{1}{8}\int (e^{-4x}+1)dx = \frac{1}{8}\left(-\frac{e^{-4x}}{4}+x\right). \end{aligned}$$

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$$\begin{aligned} v_2(x) &= -\frac{1}{4}\int e^{2x}\cosh(2x)dx = -\frac{1}{4}\int \frac{e^{2x}(e^{2x}+e^{-2x})}{2}dx \\ &= -\frac{1}{8}\int (e^{4x}+1)dx = -\frac{1}{8}\left(\frac{e^{4x}}{4}+x\right). \end{aligned}$$

Hence the general solution of the inhomogeneous equation is 2

$$\begin{aligned} y &= c_1 e^{2x} + c_2 e^{-2x} + e^{2x}\frac{1}{8}\left(-\frac{e^{-4x}}{4}+x\right) - e^{-2x}\frac{1}{8}\left(\frac{e^{4x}}{4}+x\right) \\ &= e^{2x}\left(c_1 + \frac{x}{8} - \frac{1}{32}\right) + e^{-2x}\left(c_2 - \frac{x}{8} - \frac{1}{32}\right) \end{aligned}$$

with c_1, c_2 being arbitrary constants.