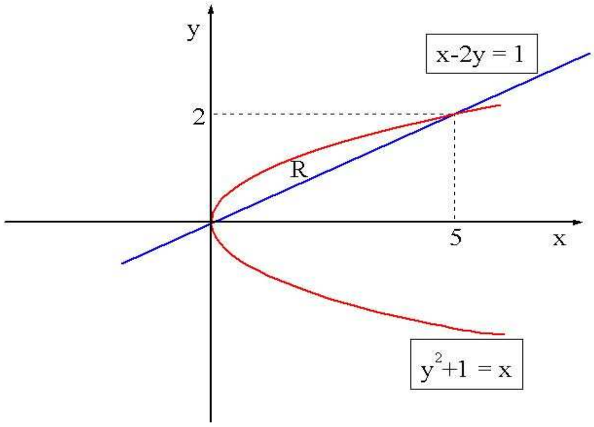
Solutions to sheet 6

1.
$$\int_{x=e}^{x=e^3}\int_{y=1}^{y=2}\frac{y}{x}dydx=\int_{x=e}^{x=e^3}\frac{1}{x}\left[\frac{y^2}{2}\right]_{y=1}^{y=2}dx=\frac{3}{2}\int_{x=e}^{x=e^3}\frac{dx}{x}=\frac{3}{2}\left[\ln x\right]_{x=e}^{x=e^3}=3.$$

2.
$$\int_{x=0}^{x=1}\int_{y=0}^{y=1}\frac{y}{1+x^2}dydx=\int_{x=0}^{x=1}\frac{1}{1+x^2}\left[\frac{y^2}{2}\right]_{y=0}^{y=1}dx=\frac{1}{2}\int_{x=0}^{x=1}\frac{dx}{1+x^2}=\frac{1}{2}\left[\tan^{-1}x\right]_{x=0}^{x=1}=\frac{\pi}{8}.$$

3. Here the tricky bit is to figure out how the integration region looks like, namely find the limits of integration in your integral. For this, we need to plot the functions $x-2y=1$ and $y^2+1=x$. The integration region R is the region which is enclosed by the two curves:

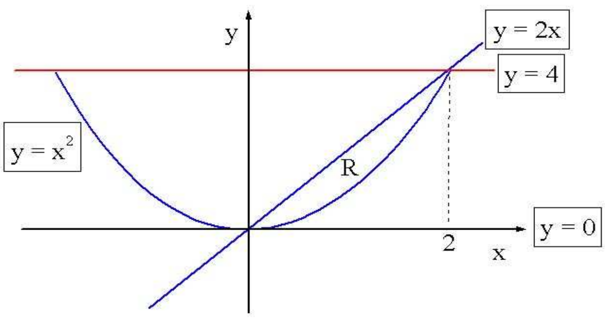


So the integral we have to compute is:

$$I=\int_{y=0}^{y=2}\int_{x=y^2+1}^{x=2y+1}\frac{1}{\sqrt{x+2y}}dxdy=\int_{y=0}^{y=2}\left[2\sqrt{x+2y}\right]_{x=y^2+1}^{x=1+2y}dy=2\int_{y=0}^{y=2}\left(\sqrt{1+4y}-\sqrt{y^2+1+2y}\right)dy$$

$$=2\int_{y=0}^{y=2}\left(\sqrt{1+4y}-y-1\right)dy=2\left[\frac{1}{6}(1+4y)^{3/2}-\frac{y^2}{2}-y\right]_{y=0}^{y=2}=2\left(\frac{1}{6}9^{3/2}-2-2-\frac{1}{6}\right)=\frac{2}{3}.$$

4. The integration region is determined by the curves $y=0, y=4, y=2x$ and $y=x^2$. Plotting all these curves we get:



So the integral we have to compute is:

$$I = \int_{y=0}^{y=4} dy \int_{x=y/2}^{x=\sqrt{y}} e^{y/x} dx = \int_{x=0}^{x=2} dx \int_{y=x^2}^{y=2x} e^{y/x} dy = \int_{x=0}^{x=2} \left[x e^{y/x} \right]_{y=x^2}^{y=2x} dx = \int_{x=0}^{x=2} x(e^x - e^2) dx,$$

and integrating by parts

$$\int_{x=0}^{x=2} x(e^x - e^2) dx = [x e^x]_{x=0}^{x=2} - \int_{x=0}^{x=2} e^x dx - e^2 \left[\frac{x^2}{2} \right]_{x=0}^{x=2} = e^2 - 1.$$

5. (a) We compute

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 10 - 12 = -2,$$

therefore $J = 2$ (remember, the Jacobian is always positive!) and $dxdy = 2dudv$.
(b) Similarly, we compute

$$\begin{vmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} \end{vmatrix} = \begin{vmatrix} \beta & \alpha \\ 2\alpha & 2\beta \end{vmatrix} = 2\beta^2 - 2\alpha^2,$$

therefore $J = 2|\beta^2 - \alpha^2|$ and $dxdy = 2|\beta^2 - \alpha^2| d\alpha d\beta$.
(c)

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0 + 1 + 1 - 0 - 0 - 0 = 2,$$

so $J = 2$ and $dxdydz = 2dudvdw$.

6. This is an exercise from the 2001 exam. A detailed solution to this problem can be downloaded from my web page:
http://www.staff.city.ac.uk/o.castro-alvaredo/teaching/solcalculus2001.pdf.

7. This is an exercise from the 2004 exam. A detailed solution to this problem can be downloaded from my web page:
http://www.staff.city.ac.uk/o.castro-alvaredo/teaching/solcalculus2004.pdf.

8. This is an exercise from the 1998 exam. A detailed solution to this problem can be downloaded from my web page:
http://www.staff.city.ac.uk/o.castro-alvaredo/teaching/solcalculus1998.pdf.