1. Use Lagrange multipliers to find the maximum and minimum distance from the curve

$$x^2 + xy + y^2 = 16$$

to the origin.

2. (a) If $y_1(x)$ and $y_2(x)$ linearly independent solutions to the linear ordinary differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

show that the Wronskian , $W(x)$, can be written as

$$W(x) = Ce^{-\int p(x)\,dx}.$$

(b) Using variation or parameters or otherwise, find the genral solution to

$$\frac{d^2y}{dx^2} + y = \sec x.$$

3. (a) Sketch the region of integration in the x–y plane for the following integral

$$I_1 = \int_0^1 \int_0^{1-x^2} \sqrt{1-y}\,\cos\left(x\sqrt{1-y}\right)\,dy\,dx.$$

Change the order of integration, and hence evaluate the integral.

(b) You are asked to find the integral of the function

$$f(x, y) = 4b^2 - x^2 - 2y^2 + bx,$$

over the region, S, in the x–y plane given by

$$a^2 \leq x^2 + y^2 \leq b^2.$$

Using polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta$$

transform the integral in x and y to one in r and θ, calculating the Jacobian of the transformation. Hence evaluate the integral

$$I_2 = \iint_S f(x, y)\,dx\,dy.$$

Turn over ...

4. Derive the p-shifting theorem for Laplace transforms:

$$\mathcal{L}(e^{ax}f(x)) = F(p-a).$$

You are required to solve

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 5r(x),$$

by the use of Laplace transforms. Show that the Laplace transform $Y(p) = \mathcal{L}(y(x))$ for general initial conditions $y(0)$ and $y'(0)$ is given by

$$Y(p) = \frac{y'(0) + (p+2)y(0) + 5R(p)}{p^2 + 2p + 5}$$

where $R(p) = \mathcal{L}(r(x))$.

For the case $y(0) = y'(0) = 0$, with

$$r(x) = x + 1,$$

find $Y(p)$.

Express $Y(p)$ in the following form:

$$Y(p) = \frac{A}{p^2} + \frac{B}{p} + \frac{C(p+1)}{(p+1)^2+4} + \frac{D}{(p+1)^2+4},$$

where A, B, C and D are to be determined. Hence find the solution $y(x)$. Check that your solution satisfies the initial conditions $y(0) = y'(0) = 0$.

[You may quote the following results:

$\mathcal{L}(f'(x)) = -f(0) + pF(p)$
$\mathcal{L}(f''(x)) = -f'(0) - pf(0) + p^2F(p)$
$\mathcal{L}(1) = 1/p$
$\mathcal{L}(x) = 1/p^2$
$\mathcal{L}(\sin(\omega x)) = \omega/(p^2 + \omega^2)$
$\mathcal{L}(\cos(\omega x)) = p/(p^2 + \omega^2)$

where $F(p) = \mathcal{L}(f(x))$ is the Laplace transform of $f(x)$.]

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