1. **Unseen** Instead of maximising $d = \sqrt{x^2 + y^2}$, maximise $x^2 + y^2$

$$\phi = x^2 + y^2 - \lambda(x^2 + +xy + y^2 - 16)$$

Solve

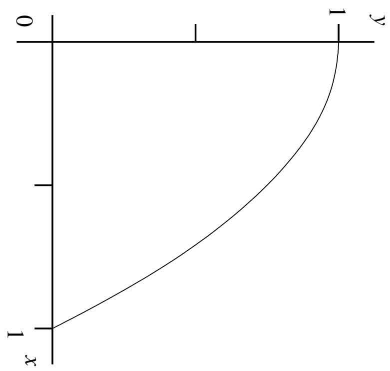
$$\phi_x = 2x - \lambda(2x + y) = 0$$
$$\phi_y = 2y - \lambda(x + 2y) = 0$$
$$x^2 + xy + y^2 = 16$$

Solutions

$$x = 4, \quad y = -4, \quad d = 4\sqrt{2}$$
$$x = -4, \quad y = +4, \quad d = 4\sqrt{2}$$
$$x = y = 4/\sqrt{3}, \quad d = 4\sqrt{2}/\sqrt{3}$$
$$x = y = -4/\sqrt{3}, \quad d = 4\sqrt{2}/\sqrt{3}$$

The first two are maxima, the last two are minima.

2. (a) Region of integration as indicated:



$$I = \int_0^1 \int_0^{\sqrt{1-y}} \sqrt{1-y}\cos\left(x\sqrt{1-y}\right)\,dx\,dy.$$
$$= \int_0^1 \left[\sin\left(x\sqrt{1-y}\right)\right]_0^{\sqrt{1-y}}\,dy.$$
$$= \int_0^1 \left[\sin(1-y) - 0\right]\,dy.$$
$$= \left[\cos(1-y)\right]_0^1 = \cos 0 - \cos 1 = 1 - \cos 1.$$

(b) **Unseen**

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

$$\iint_S f(x,y)\,dx\,dy = \int_0^{2\pi}\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_a^b \left(4b^2 - r^2\cos^2\theta - 2r^2\sin^2\theta + br\cos\theta\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[2b^2r^2 - \frac{r^4}{4}\cos^2\theta - \frac{2r^4}{4}\sin^2\theta + \frac{br^3}{3}\cos\theta\right]_0^1 d\theta$$

$$= \int_0^{2\pi}\left[2b^2(b^2-a^2) - \frac{b^4-a^4}{4}\cos^2\theta - \frac{2(b^4-a^4)}{4}\sin^2\theta + \frac{b(b^3-a^3)}{3}\cos\theta\right] d\theta$$

$$= 4\pi b^2(b^2-a^2) - \frac{\pi(b^4-a^4)}{4} - \frac{2\pi(b^4-a^4)}{4} + 0$$

$$= 4\pi b^2(b^2-a^2) - \frac{3\pi(b^4-a^4)}{4} + 0$$

3. (a) **Bookwork**

$$W(x) = y_1y_2' - y1'y_2$$

$$W'(x) = y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2'$$

$$= y_1(-py_2' - qy_2) - y_2(-py_1' - qy_1) = -p(y_1y_2' - y_2y_1') = -pW$$

$$\ln W = \int -p(x)\,dx + const$$

$$W = C\exp\left(-\int p(x)\,dx\right)$$

(b) $y_1(x) = \cos x$, $y_2(x) = \sin x$, Try $y = a(x)y_1(x) + b(x)y_2(x)$.

$$y' = a'\cos x - a\sin x + b'\sin x + b\cos x$$

Assume $a'\cos x + b'\sin x = 0$, then

$$y'' = -a'\sin x - a\cos x + b'\cos x - b\sin x$$

$$y'' + y = -a'\sin x + b'\cos x = \sec x$$

To solve

$$a'\cos x + b'\sin x = 0 \quad \text{and} \quad -a'\sin x + b'\cos x = \sec x$$

add $\sin x$ times the first to $\cos x$ times the second to get $b' = 1$. Hence $a' = -\tan x$.

$$a = \ln\cos x + c_a, \qquad b = x + c_b$$

$$y = A\cos x + B\sin x + x + \ln\cos x$$

4. **Bookwork**

$$\int_0^\infty e^{ax} f(x) e^{-px}\, dx = \int_0^\infty f(x) e^{-(p-a)x}\, dx = F(p-a).$$

Unseen

$$-y'(0) - py(0) + p^2 Y(p) + 2(-y(0) + pY(p)) + 5Y(p) = 5R(p)$$

Rearranging gives

$$Y(p) = \frac{y'(0) + (p+2)y(0) + 5R(p)}{p^2 + 2p + 5}$$

$$R(x) = \frac{1}{p^2} + \frac{1}{p} = \frac{p+1}{p^2}, \qquad Y(p) = \frac{5(p+1)}{p^2(p^2+2p+5)}$$

$$Y(p) = \frac{1}{p^2} + \frac{3}{5p} - 3\frac{(p+1)}{5((p+2)^2+4)} - \frac{8}{5(p+2)^2+4},$$

$$y(x) = x + \frac{3}{5} - \frac{3}{5}e^{-x}\cos 2x - \frac{4}{5}e^{-x}\sin 2x$$

Check

$$y(0) = 0 + \frac{3}{5} - \frac{3}{5} + 0 = 0,$$

$$y'(x) = 1 + 0 + \frac{3-8}{5}e^{-2x}\cos 2x + \frac{6+4}{5}e^{-2x}\sin 2x \quad \Rightarrow \quad y'(0) = 1 - 1 = 0$$