1. Use Lagrange multipliers to find the maximum and minimum values of

$$f(x, y) = x^3 - y^3$$

subject to the constraint

$$x^2 + y^2 = 8.$$

2. (a) Sketch the region of integration in the x–y plane for the following integral

$$I = \int_{1/2}^{2}\int_{1/2}^{1/y} \left[x^2 e^{x^2 y} - \frac{2xe^{x^2/2}}{x-2}\right] dx\, dy.$$

Change the order of integration, showing clearly what the new limits of integration should be, and hence evaluate the integral.

(b) You are required to do an integral of a function, $f(x, y)$, over an ellipse whose edge is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

A suitable change of coordinates for calculating a double integral over this elliptical region is

$$x = ar\cos\theta, \qquad y = br\sin\theta.$$

By calculating the Jacobian of the transformation, show that a double integral over this ellipse becomes

$$\iint_S f(x, y)\, dx\, dy = \int_0^{2\pi}\int_0^1 f(ar\cos\theta, br\sin\theta)\, abr\, dr\, d\theta.$$

If $f(x, y)$ given by

$$f(x, y) = 1 - x/a + y^2/b^2,$$

use the new coordinates to evaluate the integral.

Turn over ...

3. Derive the p-shifting theorem for Laplace transforms:

$$\mathcal{L}(e^{ax}f(x)) = F(p-a).$$

You are required to solve

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 8r(x),$$

by the use of Laplace transforms. Show that the Laplace transform $Y(p) = \mathcal{L}(y(x))$ for general initial conditions $y(0)$ and $y'(0)$ is given by

$$Y(p) = \frac{y'(0) + (p+4)y(0) + 8R(p)}{p^2 + 4p + 8}$$

where $R(p) = \mathcal{L}(r(x))$.

For the case $y(0) = y'(0) = 0$, with

$$r(x) = x/8,$$

find $Y(p)$.

Express $Y(p)$ in the following form:

$$Y(p) = \frac{A}{p^2} + \frac{B}{p} + \frac{C(p+2)}{(p+2)^2+4} + \frac{D}{(p+2)^2+4},$$

where A, B, C and D are to be determined. Hence find the solution $y(x)$. Check that your solution satisfies the initial conditions $y(0) = y'(0) = 0$.

[You may quote the following results:

$\mathcal{L}(f'(x)) = -f(0) + pF(p)$

$\mathcal{L}(f''(x)) = -f'(0) - pf(0) + p^2F(p)$

$\mathcal{L}(1) = 1/p$

$\mathcal{L}(x) = 1/p^2$

$\mathcal{L}(\sin(\omega x)) = \omega/(p^2 + \omega^2)$

$\mathcal{L}(\cos(\omega x)) = p/(p^2 + \omega^2)$

where $F(p) = \mathcal{L}(f(x))$ is the Laplace transform of $f(x)$.]

Turn over ...

4. By considering the variation of the Wronskian of the solutions $y_1(x)$ and $y_2(x)$ of the second order differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

find an integral representation for $y_2(x)$ in terms of $y_1(x)$.

Hence, or otherwise, given that $y_1(x) = x$ is a solution to

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0,$$

determine the general solution.

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