1. **Unseen**

$$\phi = x^3 - y^3 - \lambda(x^2 + y^2 - 8)$$

Solve

$$\phi_x = 3x^2 - \lambda x = x(3x - \lambda) = 0$$
$$\phi_y = -3y^2 - \lambda y = y(-3y - \lambda) = 0$$
$$x^2 + y^2 = 8$$

Solutions

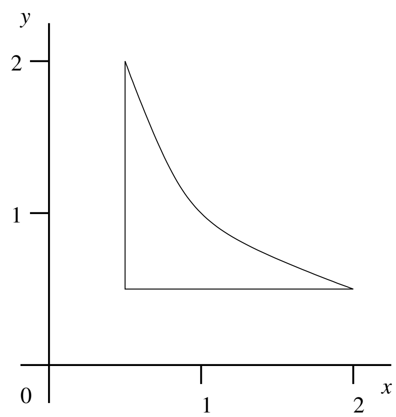
$$x = 0, \quad y = \pm 2\sqrt{2}, \quad f = \mp 16\sqrt{2}$$
$$y = 0, \quad x = \pm 2\sqrt{2}, \quad f = \pm 16\sqrt{2}$$
$$3x - \lambda = -3y - \lambda = 0 \Rightarrow x = -y, \Rightarrow x = -y = \pm 2$$
$$x = -y = 2, \quad f = 16, \quad x = -y = -2, \quad f = -16.$$

Maxima ($f = 16\sqrt{2}$) when $x = 0$, $y = -2\sqrt{2}$, or $x = 2\sqrt{2}$, $y = 0$.

Minima ($f = -16\sqrt{2}$) when $x = 0$, $y = 2\sqrt{2}$, or $x = -2\sqrt{2}$, $y = 0$.

2. **Unseen**

(a) Region of integration as indicated:



$$I = \int_{1/2}^{2}\int_{1/2}^{1/x} \left[x^2 e^{x^2 y} - \frac{2x e^{x^2/2}}{x-2} \right] dy\, dx$$
$$= \int_{1/2}^{2} \left[e^{x^2 y} - \frac{2x e^{x^2/2} y}{x-2} \right]_{1/2}^{1/x} dy\, dx$$
$$= \int_{1/2}^{2} \left[e^{x} - \frac{2e^{x^2/2}}{x-2} - e^{x^2/2} + \frac{x e^{x^2/2}}{x-2} \right] dx$$
$$= \int_{1/2}^{2} e^x\, dx = [e^x]_{1/2}^{2} = e^2 - e^{1/2}.$$

(b) **Unseen**

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta & b\sin\theta \\ -ar\sin\theta & br\cos\theta \end{vmatrix} = abr\cos^2\theta + abr\sin^2\theta = abr.$$

$$\begin{aligned}
\iint_S f(x,y)\,dx\,dy &= \int_0^{2\pi}\int_0^1 f(ar\cos\theta, br\sin\theta)\,abr\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \left(1 - r\cos\theta + r^2\sin^2\theta\right)\,abr\,dr\,d\theta \\
&= ab\int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^3}{3}\cos\theta + \frac{r^4}{4}\sin^2\theta\right]_0^1 d\theta \\
&= ab\int_0^{2\pi}\frac{1}{2} - \frac{1}{3}\cos\theta + \frac{1}{4}\sin^2\theta\,d\theta \\
&= ab\left(\pi - 0 + \frac{\pi}{4}\right) = \frac{5ab\pi}{4}.
\end{aligned}$$

3. **Bookwork**

$$\int_0^\infty e^{ax}f(x)e^{-px}\,dx = \int_0^\infty f(x)e^{-(p-a)x}\,dx = F(p-a).$$

Unseen

$$-y'(0) - py(0) + p^2Y(p) + 4(-y(0) + pY(p)) + 8Y(p) = 8R(p)$$

Rearranging gives

$$Y(p) = \frac{y'(0) + (p+4)y(0) + 8R(p)}{p^2 + 4p + 8}$$

$$R(x) = \frac{1}{8p^2}, \qquad Y(p) = \frac{1}{p^2(p^2+4p+8)}$$

$$Y(p) = \frac{1}{8p^2} - \frac{1}{16p} + \frac{(p+2)}{16((p+2)^2+4)} + \frac{0}{(p+2)^2+4},$$

$$y(x) = \frac{x}{8} - \frac{1}{16} + \frac{1}{16}e^{-2x}\cos 2x$$

Check

$$y(0) = 0 - \frac{1}{16} + \frac{1}{16} = 0,$$

$$y'(x) = \frac{1}{8} - \frac{1}{8}e^{-2x}\cos 2x - \frac{1}{8}e^{-2x}\sin 2x \quad \Rightarrow \quad y'(0) = \frac{1}{8} - \frac{1}{8} = 0$$

4. **Bookwork**

$$W(x) = y_1 y_2' - y1' y_2$$

$$W'(x) = y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2'$$

$$= y_1(-p y_2' - q y_2) - y_2(-p y_1' - q y_1) = -p(y_1 y_2' - y_2 y_1') = -pW$$

$$\ln W = \int -p(x)\, dx$$

$$W = y_1 y_2' - y_1' y_2 = \exp\left(-\int p(x)\, dx\right)$$

$$\frac{y_2'}{y_1} - \frac{y_1' y_2}{y_1^2} = \frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{\exp\left(-\int p(x)\, dx\right)}{y_1^2}$$

hence

$$y_2 = y_1 \int \frac{\exp\left(-\int p(x)\, dx\right)}{y_1^2}\, dx$$

Unseen

$$p(x) = \frac{2x}{x^2 - 1} \quad \Rightarrow \quad \int p(x)\, dx = \ln(x^2 - 1) \quad \Rightarrow \quad \exp\left(-\int p(x)\, dx\right) dx = \frac{1}{x^2 - 1}$$

$$y_2 = x \int \frac{1}{x^2(x^2 - 1)}\, dx = x \int -\frac{1}{x^2} + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}\, dx$$

$$= x\left(\frac{1}{x} + \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)\right) = 1 + \frac{x}{2}\ln\left(\frac{x-1}{x+1}\right).$$

General solution is

$$y(x) = Ay_1(x) + By_2(x) = Ax + B\left(1 + \frac{x}{2}\ln\left(\frac{x-1}{x+1}\right)\right).$$