Section A

Answer **all** questions from this section. Each question carries 8 marks.

1. Let f and g be the function defined by

$$f : [-1,\, 5] \to \mathbb{R}, \qquad g : [-2,\, 4] \to \mathbb{R};$$
$$f : x \mapsto 2x^2 - 3, \qquad g : x \longmapsto 3 - \tfrac{1}{2}x.$$

   (i) Write down which of these functions is one-one and prove that it is one-one.

   (ii) Explain why the other function is not one-one.

   (iii) Does either of the functions have an inverse? If so, find the inverse function.

2. If $f(r) = r.r!$ show that

$$f(r+1) - f(r) = r!\left(r^2 + r + 1\right).$$

   Hence find the sum of the series

$$\sum_{r=1}^{n} r!\left(r^2 + r + 1\right).$$

3. Solve the recurrence relations:

   (i) $u_{n+1} = 3u_n + 4, \qquad n = 0,\, 1,\, 2,\, ...;$

   (ii) $u_{n+2} = 2u_{n+1} + 8u_n$, $n = 0,\, 1,\, 2,\, ...$ where $u_0 = -1$ and $u_1 = 8$.

Turn over . . .

4. $\underline{M}$ is the matrix given by

$$\underline{M} = \begin{pmatrix} 3 & 1 & -3 \\ 1 & 2a & 1 \\ 0 & 2 & a \end{pmatrix}.$$

(i) Find the values of a for which $\underline{M}$ is singular.

(ii) Find the inverse matrix of $\underline{M}$ when $a = 1$.

5. The expansions of

$$\surd(1-2x) \quad \text{and} \quad 1 - x\left(1 - \frac{a}{2}x\right)^n$$

are the same as far the term in x^3.

Find the values of a and n. For these values of a and n state the common set of values of x for which both expansions are valid.

6. Vectors $\underline{a}$ and $\underline{b}$ are given by

$$\underline{a} = 2\underline{i} + \underline{k} \qquad \text{and} \qquad \underline{b} = \underline{i} + 2\underline{j} + c\underline{k}.$$

(i) If $c = 1$ find the angle between $\underline{a}$ and $\underline{b}$, giving your answer to the nearest degree.

(ii) Find the values of c for which

(a) the vector $\underline{a}$ is perpendicular to the vector $\underline{b}$,

(b) the vector $(\underline{a} \times \underline{b}) + \underline{a}$ is parallel to the vector $\underline{k}$.

Turn over . . .

Section B

Answer **two** questions from this section. Each question carries 26 marks.

7. (a) (i) Find the sixth roots of -64 and express them in Cartesian form. Show that these six roots form three pairs of conjugate complex numbers.

   (ii) Hence factorize
$$z^6 + 64$$
into three quadratic factors with real coefficients.

   (b) By considering
$$(\cos\theta + i\sin\theta)^3$$
prove that
$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

8. A set of simultaneous equations is given by

$$\begin{aligned} x - 2y + 3z &= -1, \\ 2x - 3y - kz &= -1, \\ x + ky + 6z &= -2, \end{aligned}$$

where k is an arbitrary constant.

   (i) Reduce the augmented matrix to upper triangular form by using Gaussian elimination.

   (ii) Hence find the values of k for which the equations have

   (a) no solution,

   (b) a unique solution,

   (c) an infinite number of solutions.

   (iii) Find the solutions, if any, of the set of equations, in the cases when

   (a) $k = -6$,

   (b) $k = -3$.

9. Two relations ρ_1 and ρ_2 are defined on the set $\mathbb{Z}$ of integers by

$$x \ \rho_1 \ y \ \text{ if } 3x + 5y \text{ is divisible by } 4$$

and

$$x \ \rho_2 \ y \ \text{ if } 3x^2 - y^2 \geq 0.$$

(i) For each relation, state whether or not the relation has the reflexive, symmetric and transitive properties. For each of the six cases, give either a proof or a counterexample as appropriate.

(ii) One of the two relations is an equivalence relation. State which relation this is. For this relation

(a) Show that the equivalence class containing the number 4 is the set S where

$$S = \{x \in \mathbb{Z} : x = 4m, m \in \mathbb{Z}\}.$$

(b) Find the other equivalence classes, giving your answers in a similar form to S in (a).

Internal Examiner: Mrs J. Malcolm
External Examiners: Professor S.A. Robertson
Professor D.J. Needham