Actuarial Science Mathematics Papers
Solutions to questions — Paper I

Section A

1
$$\frac{1-2x+5x^2}{(1-2x)(1+x^2)} = \frac{A}{1-2x} + \frac{Bx+C}{1+x^2}.$$

Solving we find $A = 1$, $C = 0$, $B = -2$.

[4]

$$\int \frac{1}{1-2x}\,dx + \int \frac{-2x}{1+x^2} = -\frac{1}{2}\ln(1-2x) + \int -\frac{1}{u}\,du$$

where $u = 1 + x^2$. Hence the integral equals

$$-\frac{1}{2}\ln(1-2x) - \ln(1+x^2) + C$$

[4]

2 (a) The Taylor series of f about c is given by

$$T(f,c) = \sum_{i\geq 0} \frac{f^{(i)}(c)}{i!}(x-c)^i.$$

[2]

(b) $\sqrt{(2-x)} = \sqrt{2}\sqrt{1-\frac{x}{2}}$.
Using the binomial expansion for $(1+t)^n$, with $n = \frac{1}{2}$, which is valid for $-1 < t < 1$, we get

$$T(f,c) = \sqrt{2}\left(1 + \frac{1}{2}\left(-\frac{x}{2}\right) + \frac{1}{2!}\frac{1}{2}\left(\frac{1}{2}-1\right)\left(-\frac{x}{2}\right)^2 + \cdots\right).$$

Hence the first three terms are

$$\sqrt{2}\left(1 - \frac{x}{4} - \frac{x^2}{32}\right)$$

valid for $-2 < x < 2$.

[6]

3 (a) Let $u = \ln(x)$ and $\frac{dv}{dx} = 1$. Then

$$\int \ln(x)\, dx = x\ln(x) - \int \frac{x}{x}\, dx = x\ln(x) - x + C.$$

[4]

(b) Let $u = \sin(x)$ so $\frac{du}{dx} = \cos(x)$. Then

$$\int \cot(x)\, dx = \int \frac{\cos(x)}{\sin(x)}\, dx = \ln(\sin(x)) + C.$$

[4]

4 Verify that y satisfies the given equation. **[3]**

Verify the identity by differentiating n times. **[5]**

5 For homogeneous part try e^{mx}, giving $m^2 + 4m + 5 = 0$, so $m = (-4 \pm \sqrt{-4})/2$. So complementary function is

$$y = Ae^{-2x}\cos x + Be^{-2x}\sin x.$$

For particular integral try $y = ax + b$, giving $0 + 4a + 5ax + 5b = x$, so $a = 1/5$ and $b = -4/25$. This gives the general solution

$$y = Ae^{-2x}\cos x + Be^{-2x}\sin x + x/5 - 4/25$$

[8]

6 Length given by

$$\int_0^1 \left(1 + (y')^2\right)^{1/2} dx = \int_0^1 \left(1 + (\sinh 3x)^2\right)^{1/2} dx$$

$$= \int_0^1 \cosh 3x\, dx = \left[\frac{\sinh 3x}{3}\right]_0^1 = \frac{\sinh 3}{3}.$$

[8]

Section B

7 (a) $1 - \tanh^2 = \text{sech}^2$, verification is routine. **[3]**

(b) $\tanh x = \alpha$ then

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \alpha.$$

Rearranging get

$$e^{2x} = \frac{1+\alpha}{1-\alpha} \quad \text{so} \quad x = \frac{1}{2}\ln\left(\frac{1+\alpha}{1-\alpha}\right).$$

[2]

(c) $3\tanh^2 x - 4\tanh x - 4 = 0$.
Therefore

$$\tanh x = \frac{4 \pm \sqrt{16+48}}{6} = 2 \quad \text{or} \quad -\frac{2}{3}.$$

But $|\tanh x| < 1$ so $\tanh x = -\frac{2}{3}$.
Therefore

$$x = \frac{1}{2}\ln\left(\frac{1}{5}\right).$$

[5]

(d) The given curve intersects the x-axis at 0, $\frac{\pi}{4}$ and $-\frac{3\pi}{4}$ and is negative between $-\frac{3\pi}{4}$ and $\frac{\pi}{4}$. Therefore the desired area is given by

$$\int_{-\frac{5\pi}{4}}^{-\frac{3\pi}{4}} x^2 \sin\left(x - \frac{\pi}{4}\right)\, dx - \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} x^2 \sin\left(x - \frac{\pi}{4}\right)\, dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x^2 \sin\left(x - \frac{\pi}{4}\right)\, dx.$$

[5]

We have

$$\int x^2 \sin\left(x - \frac{\pi}{4}\right)\, dx = -x^2\cos\left(x - \frac{\pi}{4}\right) + \int 2x\cos\left(x - \frac{\pi}{4}\right)$$

which equals

$$-x^2\cos\left(x - \frac{\pi}{4}\right) + 2x\sin\left(x - \frac{\pi}{4}\right) + 2\cos\left(x - \frac{\pi}{4}\right).$$

[7]

Substituting we get that the desired area is

$$\frac{6\pi}{4} + \frac{\pi^2}{16} - 2 + \frac{10\pi^2}{16} - 4 + \frac{5\pi}{4} + \frac{9\pi^2}{16} - 2$$

which equals

$$\frac{11\pi}{4} + \frac{20\pi^2}{16} - 8.$$

[4]

8 (i) If $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ or $f_{yy} > 0$ then minimum, (ii) If $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$ or $f_{yy} < 0$ then maximum, (iii) If $f_{xx}f_{yy} - f_{xy}^2 < 0$ then saddle point. Otherwise cannot tell. **[2]**

$$f_x = 4x^3 + 4xy^2 - 4(x+y) = 0$$
$$f_y = 4x^2y + 4y^3 - 4(x+y) = 0$$
$$f_x - f_y = 4x^3 - 4y^3 = 4(x-y)(x^2+xy+y^2) = 0$$

The term $x^2 + xy + y^2$ is only zero when $x = y = 0$ and this point clearly satisfies $f_x = f_y = 0$. For other stationary points we need $x = y$. Substitute into, say, $f_x = 0$ to get

$$4x^3 + 4x^3 - 4(x+x) = 8x^3 - 8x = 8x(x-1)(x+1) = 0$$

Other stationary points are $x = y = 1$ and $x = y = -1$. **[8]**

$$f_{xx} = 12x^2 + 4y^2 - 4, \qquad f_{yy} = 4x^2 + 12y^2 - 4,$$
$$f_{xy} = 8xy - 4.$$

At $x = y = 0$ we have $f_{xx}f_{yy} - f_{xy}^2 = 0$, so test fails to give any information. At $x = y = \pm 1$, $f_{xx}f_{yy} - f_{xy}^2 = 12 \times 12 - 4^2 = 128 > 0$, so minima. **[10]**

At $x = y = 0$, $f_{xx}f_{yy} - f_{xy}^2 = 0$, so test fails at origin. On $x = y$ we have $f(x,y) = 4x^4 - 8x^2$ which has a maximum at $x = y = 0$. On $x = -y$ we have $f(x,y) = 4x^4$ which has a minimum at the origin. Since $f(x,y)$ exhibits both behaviours it is neither a maximum nor a minimum at the origin. **[6]**

9 (a) Check exact derivative:

$$\frac{\partial}{\partial x}\left(y + xe^{x^2}\right) = e^{x^2} + 2x^2e^{x^2}$$

$$\frac{\partial}{\partial y}\left(x + (1+2x^2)ye^{x^2}\right) = (1+2x^2)e^{x^2}$$

[4]

Solution is

$$\frac{x^2+y^2}{2} + xye^{x^2} = C.$$

[4]

(b) For homogeneous part try $y = x^m$ giving $m(m-1) - m - 8 = x^2 - 2m - 8 = (x-4)(x+2)0$, so $m = 4$ or $m = -2$ giving complementary function

$$y = Ax^4 + Bx^{-2}.$$

[5] For particular integral try $y = ax + b$. Substitute in, giving

$$0 - ax - 8(ax + b) = x + 1$$

Hence $a = -1/9$ and $b = -1/8$, giving the general solution

$$y = Ax^4 + Bx^{-2} - x/9 - 1/8$$

[4]

(c) For the homogeneous part try $y = e^{mx}$, giving $m^2 + 2m + 1 = 0$, which has a repeated root of $m = -1$. Complementary function will be

$$y = (Ax + b)e^{-x}.$$

[4]

For particular integral try substituting in

$$y = ae^x + bx^2 e^{-x},$$

[2]

giving

$$4ae^x + 2be^{-x} = e^x + e^{-x},$$

hence $a = 1/4$ and $b = 1/2$, giving the general solution

$$y = e^x/4 + (x^2/2 + Ax + b)e^{-x}.$$

[3]